HERMITIAN TRACE METHODS
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CONTENTS

1. ALGEBRAIC K-THEORY

Let X be a sufficiently nice (say, quasi-projective) algebraic variety. Grothendieck defined the group $\mathrm{K}_0(X)$ as the abelian group generated by the set of isomorphism classes $[V]$ of vector bundles over X under the relation $[V]+[U]=[W]$ for any short exact sequence

$$0 \to V \to W \to U \to 0$$

of vector bundles on X. If $X = \mathrm{spec}(R)$ is affine then vector bundles correspond to finitely generated projective R-modules and every short exact sequence of such splits, so that the relations are of the form $[P]+[Q]=[P \oplus Q]$, in which case $\mathrm{K}_0(R)$ is just the group completion of the monoid of isomorphism classes of finitely generated projective R-modules. This innocent looking definition turned out to be quite powerful, as Grothendieck demonstrated in his celebrated generalisation of the Riemann-Roch theorem.

Hirzebruch and Atiyah realized that Grothendieck's construction also works well in the topological setting, where X is now a compact Hausdorff space and $\mathrm{K}_0(X)$ is similarly formed using complex vector bundles on X. As for rings, here again every short exact sequence splits, so we may view $\mathrm{K}_0(X)$ as a suitable group completion. This topological version also turned out to be quite powerful: using the Adams operations, which are certain natural transformations $\psi^k : \mathrm{K}_0(X) \to \mathrm{K}_0(X)$, Adams' proved that maps $S^{4n-1} \to S^{2n}$ of Hopf invariant 1 exist only in dimensions $n = 1, 2, 4$.

The importance of topological K-theory significantly rose with the discovery that K_0 can be extended to a generalized cohomology theory via Bott periodicity. More precisely, define the reduced K-group $\tilde{\mathrm{K}}(X)$ for a based topological space X as the kernel of the split projection $\mathrm{K}_0(X) \to \mathrm{K}_0(*)$. The negative reduced K-groups can be defined via the formula $\tilde{\mathrm{K}}_{-n}(X) = \tilde{\mathrm{K}}_0(\Sigma^n X)$, and they fit into a long exact sequence

$$\ldots \to \tilde{\mathrm{K}}_{-1}(X/A) \to \tilde{\mathrm{K}}_{-1}(X) \to \tilde{\mathrm{K}}_0(X/A) \to \tilde{\mathrm{K}}_0(X) \to \mathrm{K}_0(A)$$

for any sufficiently nice closed inclusion $A \subseteq X$ of based compact Hausdorff spaces. Bott then proved that

$$\tilde{\mathrm{K}}_0(X) \cong \tilde{\mathrm{K}}_0(\Sigma^2 X)$$

that is, these negative K-groups are 2-periodic, which means that we can formally extend them to positive degrees by keeping the periodicity, leading to long exact sequences stretching infinitely on both sides.

It was then natural to ask whether in the algebraic setting one can extend the group K_0 to positive or negative degrees, in a manner leading to something resembling a cohomology theory for algebraic varieties. In the case of rings, it was discovered that a well-behaved candidate for $\mathrm{K}_1(R)$ is given by the abelianization of the infinite general linear group $\mathrm{GL}(R) = \mathrm{colim}_n \mathrm{GL}_n(R)$. For example, if R is a field then this is isomorphic to the group of units R^* via the determinant. Quillen's idea was then to use the plus construction, originally introduced by Kervaira, to obtain a K-theory space. From a modern perspective, given a space X, the plus construction $X \to X_+$ can be characterized as the essentially unique acyclic map to a nilpotent space, where a map is called a cyclic if its homotopy fibres have trivial homologies with constant coefficients, and a space is called nilpotent if it is local with respect to acyclic maps. Concretely, the fundamental group $\pi_1(X_+)$ of X_+ is the quotient of $\pi_1(X)$ by its maximal perfect subgroup. In the case of $X = \mathrm{BGL}(R)$ its maximal perfect subgroup is its commutative group, which is also the subgroup generated by elementary matrices. The plus construction $\mathrm{BGL}(R)_+$ then has fundamental group $\mathrm{GL}(R)_{\mathrm{ab}} = \mathrm{K}_1(R)$, and can be viewed as a obtained from $\mathrm{BGL}(R)$ by performing the abelianization process on the space level instead of the group level. Quillen then proposed $\mathrm{K}_0(R) \times \mathrm{BGL}(R)_+$ as a definition of the K-theory space, so that the higher K-groups are given by $\mathrm{K}_n(R) = \pi_n \mathrm{BGL}(R)_+$ for $n \geq 1$.

This construction is however not directly applicable to schemes. Quillen then came up with a generalization as follows. Given a sufficiently nice algebraic variety X, Quillen's idea was to form a category $\mathrm{Q}(X)$, sometimes called the Q-construction of X, whose objects are the vector bundles V over X, and whose morphisms from V to U are spans

$$\begin{tikzcd} & W \arrow[dl, two heads] \arrow[dr, hook] & \\ V & & U \end{tikzcd}$$

whose left leg is surjective and right leg injective. Composition is defined by forming fibre products

$$\begin{tikzcd} & & W \times_U W' \arrow[dl, two heads] \arrow[dr, hook] & & \\ & W \arrow[dl, two heads] \arrow[dr, hook] & & W' \arrow[dl, two heads] \arrow[dr, hook] & \\ V & & U & & U'. \end{tikzcd}$$

Quillen then defined the algebraic K-theory *space* of X to be

$$\mathcal{K}(X) := \Omega |\, \mathrm{Q}(X)|,$$

that is, the loops on the geometric realization of (the nerve of) the category $\mathrm{Q}(X)$.

1.0.1. **Proposition.** *There is a natural isomorphism between $\pi_0 \mathcal{K}(X) = \pi_1 |\, \mathrm{Q}(X)|$ and the group $\mathrm{K}_0(X)$ defined above.*

The higher K-groups are then defined as the homotopy groups

$$\mathrm{K}_n(X) := \pi_n \Omega |\, \mathrm{Q}(X)| = \pi_{n+1} |\, \mathrm{Q}(X)|.$$

Quillen then also showed that for rings there is a natural equivalence $\Omega |\, \mathrm{Q}(\mathrm{spec}(R))| \simeq \mathrm{K}_0(R) \times \mathrm{BGL}(R)_+$. In particular, $\mathrm{K}_1(\mathrm{spec}(R))$ is the abelianization of $\mathrm{GL}(R)$.

Proof. Let Set be the category of sets, and write $\mathrm{Fun}^{\simeq}(\mathrm{Q}(X), \mathrm{Set}) \subseteq \mathrm{Fun}(\mathrm{Q}(X), \mathrm{Set})$ for the full subcategory spanned by the functors which invert all arrows. By covering space theory it suffices to prove an equivalence of categories

$$\mathrm{Fun}^{\simeq}(\mathrm{Q}(X), \mathrm{Set}) \simeq \mathrm{K}_0(X) - \mathrm{Set}$$

compatible with the forgetful functor to Set, where on the right hand side we have the category of sets with $\mathrm{K}_0(X)$-action. We construct an explicit pair of inverse equivalences as follows. Let $\mathrm{Vect}^{\mathrm{inc}}(X) \hookrightarrow \mathrm{Q}(X)$ be the wide subcategory consisting of the purely forward spans. Then $\mathrm{Vect}^{\mathrm{inc}}(X)$ has an initial object and so the functor $\mathrm{const} : \mathrm{Set} \to \mathrm{Fun}(\mathrm{Vect}^{\mathrm{inc}}, \mathrm{Set})$ obtained by restriction along $\mathrm{Vect}^{\mathrm{inc}} \to *$, is an equivalence of categories. We hence may as well construct the inverse equivalences between

$$\mathrm{Fun}^{\simeq}(\mathrm{Q}(X), \mathrm{Set}) \times_{\mathrm{Fun}(\mathrm{Vect}^{\mathrm{inc}}, \mathrm{Set})} \mathrm{Set} \quad \text{and} \quad \mathrm{K}_0(X) - \mathrm{Set}.$$

Now given a set S with an action of $\mathrm{K}_0(X)$ we define $F_S\colon\ \mathrm{Q}(X) \to \mathrm{Set}$ by $F_S(V) = S$ for every V with functoriality sending a span

$$\begin{tikzcd} & W \arrow[dl, two heads, "p"'] \arrow[dr, hook] & \\ V & & U \end{tikzcd}$$

to the action of $[\ker(p)] \in \mathrm{K}_0(X)$ on S. This is indeed compatible with composition because the kernel of the projection in a composite span sits in an exact sequence with the kernels of the projections of the two constituents. We also note that the restriction of F_S to $\mathrm{Vect}^{\mathrm{inc}}(X)$ is constant on S by construction. In the other direction, given a functor $F\colon\ \mathrm{Q}(X) \to \mathrm{Set}$, a set S, and a natural isomorphism $\eta\colon\ F|_{\mathrm{Vect}^{\mathrm{inc}}(X)} \cong \mathrm{const}(S)$, we define an action of $\mathrm{K}_0(X)$ on S via the defining universal property of the group $\mathrm{K}_0(X)$, that is, we define an action of the monoid $\pi_0\mathrm{Vect}(X)^{\simeq}$ of isomorphism classes of vector bundles and then show that this action is compatible with short exact sequences. Specifically, we define the action of $[V] \in \pi_0\mathrm{Vect}(X)^{\simeq}$ to be the action of the purely backward span

$$\begin{tikzcd} & V \arrow[dl, two heads] \arrow[dr, equal] & \\ 0 & & V \end{tikzcd}$$

using that both $F(0)$ and $F(V)$ are identifies with S via η. The compatiblity with short exact sequences $0 \to V \to W \to U \to 0$ is then obtained by considering the two composable pairs of spans:

$$\begin{tikzcd}[column sep=small] & & W \arrow[dl, two heads] \arrow[dr, equal] & & \\ & U \arrow[dl, two heads] \arrow[dr, equal] & & W \arrow[dl, two heads] \arrow[dr, equal] & \\ 0 & & U & & W \end{tikzcd} \quad \text{and} \quad \begin{tikzcd}[column sep=small] & & V \arrow[dl, two heads] \arrow[dr, hook] & & \\ & 0 \arrow[dl, equal] \arrow[dr, hook] & & W \arrow[dl, two heads] \arrow[dr, equal] & \\ 0 & & U & & W. \end{tikzcd}$$

The first shows that the action of $[W] \in \pi_0\mathrm{Vect}(X)^{\simeq}$ is equal to the action of $[U]$ plus the action of the $U \twoheadleftarrow W = W$, and the latter is equal to the action of $[V]$ by the second composable pair. It is then straightforward to check the functors just defined form an inverse pair of equivalences of categories.

□

For commutative rings and schemes, the tensor product of projective modules/vector bundles induces a graded ring structure on the K-groups.

1.0.2. **Example.** By Suslin's work the algebraic K-theory of all algebraically closed field of a fixed characteristic is the same up to uniquely divisible groups. In particular, if k is an algebraically closed field of characteristic 0 then $\mathrm{K}_n(X)$ is uniquely divisible for n even and a direct sum of a uniquely divisible group and $\mathbb{Q}/\mathbb{Z}$ for n even (in $\mathrm{K}_1(X)$, this $\mathbb{Q}/\mathbb{Z}$ corresponds to the roots of unity in k^*). If k has positive characteristic then the same holds in even degrees and in odd degrees we have $\mathbb{Q}/\mathbb{Z}[1/p]$ instead, which again corresponds to the roots of unit in degree 1. One can also define K-groups with coefficients $\mathrm{K}_i(k, \mathbb{Z}/m)$, which sit in a long exact sequence with the K-groups $\mathrm{K}_i(k)$ and multiplication by m map. If m is prime to the characteristic then by the above these groups are $\mathbb{Z}/m$ in even degrees and 0 elsewhere, and as a graded ring are generated by a single generator in degree 2. In particular, they behave like the mod m topological K-theory of the point.

1.0.3. **Example.** By the work of Illusie, Geisser and Levine, if k is a separably, but not necessarily algebraically, closed field of characteristic $p > 0$, then the K-groups are still p-torsion free, but can have non-trivial p-cotorsion. However, the graded ring $\mathrm{K}_*(k)/p$ is generated by elements in degree 1, and $\mathrm{K}_1(k)/p = k^*/(k^*)^p$ can be realized as a subgroup of the module of absolute Kahler differentials Ω_k by the association $u \mapsto d\log(u) := \frac{du}{u}$ for $u \in k^*$ which vanishes on p-powers. Similarly, $\mathrm{K}_i(k)/p$ can be embedded in $\Lambda^i_k\Omega_k$ as the submodule spanned by $d\log(u_1) \wedge ... \wedge d\log(u_n)$ for $u_1, ..., u_n \in k^*$. In particular, if k is a finite extension of degree p^d of the subfield k^p of p-powers then Ω_k is d-dimensional as a k-vector space and so $\mathrm{K}_i(k)/p$ vanishes for $i > d$.

Quillen's definition of algebraic K-theory via the Q-construction only depends on a certain category associated to X, namely, the category of vector bundles. More precisely, it also makes use of the fact that we have two distinguished classes of morphisms, namely, surjections and inclusions, so that we can speak of short exact sequences. The properties that these classes of morphisms satisfy can be axiomatize via the notion of an exact category:

1.0.4. **Definition.** An *exact category* is a category $\mathcal{E}$ together with two classes of morphisms, called inclusions and surjections, satisfying the following axioms:

i) The category $\mathcal{E}$ is additive.
ii) Pullback along surjections exist and are again surjections.
iii) Pushout along inclusions exist and are again inclusions.
iv) Given a commutative square

$$\begin{tikzcd} x \arrow[r, "f'"] \arrow[d, "g"] & y \arrow[d, "g'"] \\ z \arrow[r, "f"] & w \end{tikzcd}$$

the following are equivalent:
- f is an inclusion, g' is a surjection and the square is pullback.
- f' is an inclusion g is a surjection and the square is pushout.

A functor $\mathcal{E} \to \mathcal{E}'$ among exact categories then preserves inclusions and pushouts along inclusions if and only if it preserves surjections and pullback along surjections, and we call a functor satisfying these equivalent conditions *exact*.

In particular, in any exact category every surjection $y \twoheadrightarrow z$ admits a fibre and every inclusion admits a cofibre. In addition, a zero-composite sequence

$$x \rightarrowtail y \twoheadrightarrow z$$

consisting of an inclusion followed by a projection is a fibre sequence if and only if it is a cofibre sequence. Such sequences are then called *exact sequences*. The exact structure on $\mathcal{E}$ can then be completely recovered from the collection of exact sequences.

1.0.5. **Example.**

i) If X is a scheme then the category $\mathrm{Vect}(X)$ of vector bundles over X admits an exact structure where the inclusions and surjections are defined fibre-wise over X.
ii) If $\mathcal{A}$ is an additive category then we can put an exact structure on it by declaring that the inclusions are the summand inclusions $x \to x \oplus y$ and the surjections are the summand projections $x \oplus y \to x$. This is the minimal exact structure we can put on $\mathcal{A}$.
iii) If $\mathcal{A}$ is an abelian category then we can put an exact structure on it by declaring that the inclusions are the monomorphisms and the surjections are the epimorphisms.

Given an exact category $\mathcal{E}$ one can define its Q-construction $\mathrm{Q}(\mathcal{E})$ just as in the case of vector bundles using spans with one leg surjection and one leg inclusion, and similarly define its K-theory space by

$$\mathcal{K}(\mathcal{E}) := \Omega|\,\mathrm{Q}(\mathcal{E})|.$$

This very abstract definition feels quite far from the geometry of manifolds. How can one prove that it exhibits behaviour of the type we can expect from a cohomology theory for varieties? In other words, where does the formal theory see the geometry?

To approach this question Quillen proved that his definition of algebraic K-theory satisfies the following properties:

i) Group-likeness
ii) Additivity.
iii) Resolution.
iv) Localisation for abelian categories.

Let us now explain each of these properties in a bit more detail. Recall first that geometric realizations preserve products, and hence the association $\mathcal{E} \mapsto \mathcal{K}(\mathcal{E})$ preserves products. Since any exact category $\mathcal{E}$ is additive by definition, it admits a direct sum functor $- \oplus - : \mathcal{E} \times \mathcal{E} \to \mathcal{E}$, which is simultaneously the categorical product and coproduct. By product preservation this induces an operation $\mathcal{K}(\mathcal{E}) \times \mathcal{K}(\mathcal{E}) \to \mathcal{K}(\mathcal{E})$ on the level of K-theory. The group-like property above then says that multiplication by any point is an invertible operation. In other words, $\mathcal{K}(\mathcal{E})$ has the structure of an H-space. But this H-space structure can be further refined. Since $\oplus$ is a symmetric monoidal structure the product we get on $\mathcal{K}(\mathcal{E})$ is associative and commutative up to homotopy. In fact, instead of considering associativity and commutativity up to homotopy, we can also keep track of all of these homotopies, endowing $\mathcal{K}(\mathcal{E})$ with the structure of an E_∞-group. To avoid confusion, note that E_∞-group structure cannot always be rigidified into a strictly commutative operation.

For additivity, consider the following construction. Given an exact category $\mathcal{E}$, we may form the exact category $\mathrm{Ex}(\mathcal{E})$ whose objects are the exact sequences

$$M \rightarrowtail N \twoheadrightarrow P$$

in $\mathcal{E}$ and whose exact structure is such that the exact sequences in $\mathrm{Ex}(\mathcal{E})$ are the pointwise exact sequences

$$\begin{tikzcd}
M \arrow[r, hook] \arrow[d, hook] & N \arrow[r, two heads] \arrow[d, hook] & P \arrow[d, hook] \\
M' \arrow[r, hook] \arrow[d, two heads] & N' \arrow[r, two heads] \arrow[d, two heads] & P' \arrow[d, two heads] \\
M'' \arrow[r, hook] & N'' \arrow[r, two heads] & P''
\end{tikzcd}$$

among exact sequences. The additivity property of algebraic K-theory can then be formulated as follows:

1.0.6. **Theorem** (Additivity)**.** *The exact functor*

$$\mathrm{Ex}(\mathcal{E}) \to \mathcal{E} \times \mathcal{E} \qquad [M \rightarrowtail N \twoheadrightarrow P] \mapsto (M, P)$$

induces an equivalence

$$\mathcal{K}(\mathrm{Ex}(\mathcal{E})) \simeq \mathcal{K}(\mathcal{E}) \times \mathcal{K}(\mathcal{E})$$

Now from additivity alone one does not see much geometry. A more powerful property of algebraic K-theory is localisation, which Quillen only proved in the beginning for abelian categories (considered as special instances of exact categories, see Example 1.0.5). Now this might seem to take us farther from geometry, since the algebraic K-theory of a variety was defined in terms of its vector bundles, and that is generally not an abelian category. We can of course consider other categories associated to X, for example, the category of coherent complexes is abelian, and we can take its algebraic K-theory. This is indeed, an interesting invariant, known as G-theory of X. To give some relation between these two constructions, Quillen proved his resolution theorem:

1.0.7. **Theorem** (Resolution)**.** *Let $\mathcal{A} \subseteq \mathcal{B}$ be an extension closed inclusion of exact categories. Suppose the following hold:*

i) If $p : P \twoheadrightarrow Q$ is a surjection in $\mathcal{B}$ and P, Q belong to $\mathcal{A}$ then the $\mathrm{fib}[P \twoheadrightarrow Q]$ also belongs to $\mathcal{A}$.

ii) Every object $M \in \mathcal{B}$ admits a finite resolution by objects in $\mathcal{A}$, that is, we have a sequence of objects $M_0, ..., M_n \in \mathcal{B}$ with $M_0 = 0$ and $M_n = M$ which participate in exact sequences

$$M_{i-1} \rightarrowtail P_i \twoheadrightarrow M_i$$

with $P_i \in \mathcal{E}$ for $i = 1, ..., n$.

Then the induced map $\mathcal{K}(\mathcal{A}) \to \mathcal{K}(\mathcal{B})$ is an equivalence.

Here an extension closed inclusion of exact categories means that $\mathcal{A}$ is an extension closed subcategory of $\mathcal{B}$ and the exact structure on $\mathcal{A}$ is inherited from that of $\mathcal{B}$ in the sense that the exact sequences are exactly those which are sent to exact sequences in $\mathcal{B}$. The geometric example to keep in mind for the resolution theorem is when X is a smooth variety (or, more generally, a regular noetherian separated scheme), $\mathcal{A} =$

Vect(X) and $\mathcal{B} = \mathrm{Coh}(X)$, since for such varieties every coherent sheaf admits a finite resolution by vector bundles. This means that K-theory coincides with G-theory for these varieties, in which case we may as well work with the abelian category of coherent sheaves. Let us now explain what Quillen's localisation theorem says.

Recall that if $\mathcal{B}$ is an abelian category then an abelian subcategory $\mathcal{A} \subseteq \mathcal{B}$ is called a *Serre subcategory* if it is closed under extensions, subobjects, and quotients. A typical example of a Serre subcategory is the subcategory of finite abelian groups inside all finitely generated abelian groups. Given a Serre subcategory $\mathcal{A} \subseteq \mathcal{B}$, let us a say that a map $M \to N$ in $\mathcal{B}$ is an equivalence mod $\mathcal{A}$ if its kernel and cokernel are in $\mathcal{A}$. The Serre quotient $\mathcal{B}/\mathcal{A}$ is then defined as the abelian category whose objects are those of $\mathcal{B}$, and such that for $M, N \in \mathcal{B}$ we have

$$\mathrm{Hom}_{\mathcal{B}/\mathcal{A}}(M, N) = \operatorname*{colim}_{M' \rightarrowtail M, N \twoheadrightarrow N'} (M', N')$$

where the colimit is taken over all subobjects $M' \rightarrowtail M$ whose cokernel lies in $\mathcal{A}$ and quotients $N \twoheadrightarrow N'$ whose kernel lies in $\mathcal{A}$. This is again an abelian category and the sequence

$$\mathcal{A} \to \mathcal{B} \to \mathcal{B}/\mathcal{A}$$

is a fibre-cofibre sequence of abelian categories, which we will call the associated Serre sequence.

1.0.8. **Theorem** (Localisation)**.** *For every Serre sequence $\mathcal{A} \to \mathcal{B} \to \mathcal{B}/\mathcal{A}$ of abelian categories the sequence*

$$\mathcal{K}(\mathcal{A}) \to \mathcal{K}(\mathcal{B}) \to \mathcal{K}(\mathcal{B}/\mathcal{A})$$

is a fibre sequence of spaces and the second map is surjective on π_0.

Let us say a couple of words about the proof of the localisation theorem. Quillen applies his Theorem B, which says the following: if $\mathcal{I} \to \mathcal{J}$ is a functor such that for every map $f: j \to j'$ in $\mathcal{J}$ the induced functor

$$\mathcal{I}_{j/} \to \mathcal{I}_{j'/}$$

is a weak homotopy equivalence, then for any object $j \in \mathcal{J}$ the square

$$\begin{tikzcd} {|\mathcal{I}_{j/}|} \arrow[r] \arrow[d] & {|\mathcal{J}_{j/}| \simeq *} \arrow[d] \\ {|\mathcal{I}|} \arrow[r] & {|\mathcal{J}|} \end{tikzcd}$$

is cartesian. One now applies this theorem in the case of the induced functor

$$\mathrm{Q}(\mathcal{B}) \to \mathrm{Q}(\mathcal{B}/\mathcal{A}).$$

To prove the localisation theorem it is then enough to prove the following two claims:

i) For every morphism $\alpha : V \twoheadleftarrow W \rightarrowtail U$ in $\mathrm{Q}(\mathcal{B}/\mathcal{A})$, the induced functor

$$\mathrm{Q}(\mathcal{B})_{U/} \to \mathrm{Q}(\mathcal{B})_{V/}$$

is a weak homotopy equivalence.

ii) The inclusion $\mathrm{Q}(\mathcal{A}) \to \mathrm{Q}(\mathcal{B})_{0/}$ is a weak homotopy equivalence.

For the first claim, note that every span factors as a composite of a purely backward span followed by a purely forward span, and so it suffices to prove the claim for α one of these two classes. Replacing all categories by their opposites, one may then reduce to showing this only for purely forward spans $W = W \rightarrowtail U$. Finally, by 2-out-of-3 it suffices to prove this for the purely forward span of the form $0 = 0 \rightarrowtail U$. Now the main idea of the proof is to replace $\mathrm{Q}(\mathcal{B})_{V/}$ by a more convenient model which does not make precise reference to morphisms in $\mathcal{B}/\mathcal{A}$ (which are somewhat less explicit). Specifically, given $M \in \mathcal{B}$ with $p(M) = V$, one considers the category $\mathcal{E}_M$ which one can define purely in terms of $\mathcal{B}$: the objects of $\mathcal{E}_M$ are arrows $f : N \to M$ in $\mathcal{B}$ whose kernel and cokernel are in $\mathcal{A}$, and its morphisms are spans

$$\begin{tikzcd} N \arrow[dr] & N' \arrow[l, two heads] \arrow[r, hook] \arrow[d] & N'' \arrow[dl] \\ & M & \end{tikzcd}$$

in $\mathcal{B}_{/M}$ with left leg surjective with kernel in $\mathcal{A}$ and right leg injective with cokernel in $\mathcal{A}$. Now since $p(f): p(N) \to V$ is an isomorphism in $\mathcal{B}/\mathcal{A}$ it is in particular surjective and we may view it as a purely backward span from V to $p(N)$. Sending (N, f) to $(N, p(f))$ then defines a functor $\mathcal{E}_M \to \mathrm{Q}(\mathcal{B})_{V/}$, and Quillen shows that this functor is a weak homotopy equivalence. Note that this already implies (2) above by taking $M = 0$. Now the advantage of the above model is that $f: M' \to M$ is a map in $\mathcal{B}$ such that $p(f)$ is an inclusion in $\mathcal{B}/\mathcal{A}$ then the contravariant functoriality of $\mathrm{Q}(\mathcal{B})_{V/}$ in $p(f)$ (viewed as a purely forward span) can be modelled via the funtor $\mathcal{E}_M \to \mathcal{E}_{M'}$ induced via pullback along f. To finish the proof one then shows that pullback along $0 \rightarrowtail M$ induces a weak homotopy equivalence

$$\mathcal{E}_M \to \mathcal{E}_0 = \mathrm{Q}(\mathcal{A}).$$

For this, one replaces $\mathcal{E}_M$ by its full subcategory $\mathcal{E}'_M \subseteq \mathcal{E}_M$ spanned by the surjections $N \twoheadrightarrow M$, and proves that $\mathcal{E}'_M \to \mathcal{E}_M$ and $\mathcal{E}'_M \to \mathrm{Q}^h(\mathcal{A})$ are both homotopy equivalences. For the second, one verifies that the functor in question is co-initial, while for the first on uses another application of Quillen's Theorem B.

The relation between the localisation theorem and geometry is that if $U \subseteq X$ is an open embedding of varieties then $\mathrm{Coh}(U)$ is a Serre quotient of $\mathrm{Coh}(X)$, with kernel the Serre subcategory $\mathrm{Coh}_Z(X) \subseteq \mathrm{Coh}(X)$ given by the coherent sheaves supported on the reduced complement $X \setminus U$ of U. If we now cover X by two open subsets $X = U \cup V$ with intersection $W = V \cap U$ then $\mathrm{Coh}_{X\setminus U}(X) = \mathrm{Coh}_{V\setminus W}(W)$, and so the localisation theorem implies that the square

$$\begin{tikzcd} \mathcal{K}(\mathrm{Coh}(X)) \arrow[r] \arrow[d] & \mathcal{K}(\mathrm{Coh}(U)) \arrow[d] \\ \mathcal{K}(\mathrm{Coh}(V)) \arrow[r] & \mathcal{K}(\mathrm{Coh}(W)) \end{tikzcd}$$

is cartesian. This is a fundamentally geometric statement (analogous to the Mayer-Vietoris property for topological spaces, which is one of the key properties of a generalized cohomology theory), which can be reformulated as Zariski descent. A-priori these K-theories are not the K-theories of the corresponding varieties, but rather their G-theories, but if X is a smooth variety over a field then these coincide with the corresponding K-theory spaces. We will see later that the K-theory also satisfies Mayer-Vietoris for non-smooth varieties except that the maps $\mathrm{K}_0(X) \to \mathrm{K}_0(U)$ and generally not surjective on K_0.

2. From exact to stable categories

Quillen's insight that algebraic K-theory is an invariant of categories rather than rings becomes significantly more powerful when categories are replaced with higher categories. To explain how this is done, let us begin with a crash course in ∞-categories, following Joyal and Lurie.

Recall that for $0 \leq i \leq n$ the i'th *horn* of Δ^n is the subsimplicial set $\Lambda^n_i \subseteq \Delta^n$ spanned by all the $(n-1)$-faces of Δ^n which contain the vertex i.

2.0.1. **Definition.** An ∞-*category* is a simplicial set $\mathcal{C}$ with the following property: for every $0 < i < n$ the dotted extension exists in any diagram of the form

(1)
$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d, hook] & \mathcal{C} \\ \Delta^n \arrow[ur, dotted] & \end{tikzcd}$$

Given an ∞-category $\mathcal{C}$ we will call the vertices of $\mathcal{C}$ its *objects*, and if we have an edge $e: \Delta^1 \longrightarrow \mathcal{C}$ then we will call it a *morphism* from $x := e|_{\Delta^{\{0\}}}$ to $y := e|_{\Delta^{\{1\}}}$, in which case we will also use diagramatic notation and write this edge as an arrow $x \longrightarrow y$. Given an object $x \in \mathcal{C}$ we will consider the degenerate edge $s(x)$ on x as the *identity* $\mathrm{id}_x : x \longrightarrow x$. Given a 2-simplex $\sigma : \Delta^2 \longrightarrow \mathcal{C}$ we will depict it diagramatically as

(2)
$$\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "h"] & & z \end{tikzcd}$$

where x, y, z are the vertices obtained by restricting σ to $\Delta^{\{0\}}, \Delta^{\{1\}}$ and $\Delta^{\{2\}}$ respectively, and f, g, h are the morphisms obtained by restricting σ to $\Delta^{\{0,1\}}, \Delta^{\{1,2\}}$ and $\Delta^{\{0,2\}}$ respectively. We can think of σ as designating a *homotopy* from $g \circ f$ to h. However, it is important to note that in an ∞-category there isn't a specific edge which *is* $g \circ f$. Instead, we may consider any triangle of the form (2) to be *exhibiting h as the composition of f and g*. In particular, there could be two different triangles $\sigma, \sigma' : \Delta^2 \longrightarrow \mathcal{C}$ of the form (2), with different edges $\sigma|_{\Delta^{\{0,2\}}} = h \neq h' = \sigma'|_{\Delta^{\{0,2\}}}$. However, in this case the edges h and h' will be *homotopic* in a suitable sense: indeed, let $\tau : \Delta^2 \longrightarrow \Delta^1 \longrightarrow \mathcal{C}$ be the degenerate triangle such that $\tau|_{\Delta^{\{1,2\}}} = \tau|_{\Delta^{\{0,2\}}} = f$ and $\tau|_{\Delta^{\{0,1\}}} = s(x)$ is degenerate on x. The triangles σ, σ' and τ then determine a map $\rho : \Lambda^2_3 \longrightarrow \mathcal{C}$ such that $\rho|_{\Delta^{\{0,1,2\}}} = \tau$, $\rho|_{\Delta^{\{0,2,3\}}} = \sigma$ and $\rho|_{\Delta^{\{1,2,3\}}} = \sigma'$. Since $\mathcal{C}$ is an ∞-category the map ρ extends to a map $\widetilde{\rho} : \Delta^3 \longrightarrow \mathcal{C}$. The 2-simplex $\widetilde{\rho}|_{\Delta^{\{0,1,3\}}}$ then determines a diagram in $\mathcal{C}$ of the form

(3)
$$\begin{tikzcd} & x \arrow[dr, "h'"] & \\ x \arrow[ur, equal, "\mathrm{id}"] \arrow[rr, "h"] & & z \end{tikzcd}$$

which we will consider as a *homotopy* from h' to h. In this sense, while the composition of two morphisms in an ∞-category is not strictly speaking uniquely defined, it is uniquely defined up to homotopy. Elaborating on this argument one can show that the collection of compositions of f and g can be organized into a simplicial set which is a *contractible Kan complex* . We may thus say that composition of two morphisms in an ∞-category is *essentially defined*. This essentially defined composition is then associative up to homotopy. In particular, we can define an ordinary category $\mathrm{Ho}(\mathcal{C})$ whose objects are the vertices of $\mathcal{C}$ and such that $\mathrm{Hom}_{\mathrm{Ho}(\mathcal{C})}(x, y)$ is the set of *homotopy classes* of arrows from x to y. This category is known as the *homotopy category* of $\mathcal{C}$. There is an evident map $\mathcal{C} \longrightarrow \mathrm{N}(\mathrm{Ho}(\mathcal{C}))$ which sends each arrow to its homotopy class.

Below are some of the common manners in which ∞-categories arise:

2.0.2. **Example.**

i) The nerve of any ordinary category is an ∞-category. These are exactly the ∞-categories for which the lift in (1) is always unique.
ii) The singular complex of any topological space is an ∞-category. More generally, any Kan complex (i.e., the simplicial sets where the lift in (1) also exists when $i = 0, n$) is an ∞-category. These are exactly the ∞-categories in which every arrow is invertible (see below).
iii) If we start from a topological category or from a category enriched in Kan complexes then we can use a certain coherent nerve construction to obtain an ∞-category. This allows one to construct, for example, the ∞-category $\mathcal{S}$ of spaces (as the coherent nerve of the category of Kan complexes, or CW complexes) or the ∞-category Cat_∞ of ∞-categories.
iv) Similarly, any dg-category (that is, a category enriched in chain complexes) can be turned in to an ∞-category using the dg-nerve construction. For example, given an abelian category $\mathcal{A}$ with enough projectives one can construct the bounded below derived category $\mathcal{D}^+(\mathcal{A})$ as the dg-nerve of the dg-category of bounded below complexes of projectives, and similarly if $\mathcal{A}$ has enough injectives we can construct the bounded above derived category $\mathcal{D}^-(\mathcal{A})$ by using bounded above complexes of injectives.
v) If $\mathcal{C}$ is an ∞-category and K is a simplicial set then the simplicial set $\mathrm{Fun}(K, \mathcal{C})$ whose n-simplices are by definition maps $\Delta^n \times K \to \mathcal{C}$, is again an ∞-category. We call $\mathrm{Fun}(K, \mathcal{C})$ the ∞-category of K-indexed diagrams in $\mathcal{C}$. If K is also an ∞-category then we also call this the functor category from K to $\mathcal{C}$.

2.0.3. **Definition.** Let $\mathcal{C}$ be an ∞-category and $f : x \longrightarrow y$ an arrow in $\mathcal{C}$. We will say that f is *invertible* if there exists an arrow $g: y \longrightarrow x$ and triangles as in (2) exhibiting id_x as the composition of f and g and id_y as the composition of g and f. In this case we will also say that f is an *equivalence* in $\mathcal{C}$ from x to y, and that x and y are equivalent. Note that, essentially by the definition of $\mathrm{Ho}(\mathcal{C})$, an arrow is invertible if and only if it maps to an isomorphism in $\mathrm{Ho}(\mathcal{C})$, and two objects are equivalent if and only if they become isomorphic in $\mathrm{Ho}(\mathcal{C})$.

We will say that an ∞-category $\mathcal{C}$ is an ∞-*groupoid* if every arrow in $\mathcal{C}$ is invertible.

2.0.4. **Theorem** (Joyal)**.** *Let $\mathcal{C}$ be an ∞-category. Then $\mathcal{C}$ is an ∞-groupoid if and only if it is a Kan complex, that is, if and only if the dotted extension exists in any diagram of the form* (1) *where $0 \leq i \leq n$.*

2.0.5. **Definition.** Given an ∞-category $\mathcal{C}$ we will denote by $\mathcal{C}^{\simeq} \subseteq \mathcal{C}$ the subsimplicial set such that an n-simplex $\sigma : \Delta^n \longrightarrow \mathcal{C}$ belongs to $\mathcal{C}^{\simeq}$ if and only if the edge $\sigma|_{\Delta^{\{i,j\}}}$ is invertible for every $i, j \in \{0, ..., n\}$. Then $\mathcal{C}^{\simeq}$ is also an ∞-category, and every morphism in $\mathcal{C}$ is invertible, i.e., $\mathcal{C}^{\simeq}$ is an ∞-groupoid. By construction, $\mathcal{C}^{\simeq}$ is the maximal sub-∞-groupoid of $\mathcal{C}$, that is, it contains any other subsimplicial set of $\mathcal{C}$ which is an ∞-groupoid.

2.0.6. **Definition.** Given an ∞-category $\mathcal{C}$ and two objects $x, y \in \mathcal{C}$, the fibre product

$$\mathrm{Map}_{\mathcal{C}}(x, y) := \mathrm{Fun}(\Delta^1, \mathcal{C}) \times_{\mathrm{Fun}(\Delta^{\{0\}} \amalg \Delta^{\{1\}})} \{(x, y)\}$$

is an ∞-groupoid, hence a Kan complex. We will refer to it as the mapping space from X to Y.

2.0.7. **Definition.** Let $\mathcal{C}$ be an ∞-category and $x \in \mathcal{C}$ an object. We will say that x is *final* if $\mathrm{Map}_{\mathcal{C}}(y, x)$ is contractible for every $y \in \mathcal{C}$. Dually, we will say that x is *initial* if $\mathrm{Map}_{\mathcal{C}}(x, y) \simeq *$ for every $y \in \mathcal{C}$.

Limits and colimits in ∞-categories can be defined in terms of finality/initiality of cones.

2.0.8. **Definition.** Let K be a simplicial set. We define the right cone $K^{\triangleright}$ to be the simplicial sets whose n-simplices are pairs (ρ, σ) where $\rho: \Delta^n \to \Delta^1$ is a map among simplices and $\sigma: \Delta^n \times_{\Delta^1} \Delta^{\{0\}} \to K$ is a map of simplicial sets (where we note that the domain of σ is either empty or a simplex). Dually, the left cone $K^{\triangleleft}$ is defined in a similar manner by replacing $\Delta^{\{0\}}$ with $\Delta^{\{1\}}$ in the definition of the domain of σ.

2.0.9. **Definition.** Let $\mathcal{C}$ be an ∞-category and $p: K \longrightarrow \mathcal{C}$ a diagram in $\mathcal{C}$. We will say that a left cone $\overline{p}: K^{\triangleleft} \longrightarrow \mathcal{C}$ is a *limit cone* if it is terminal as an object of $\mathrm{Fun}(K^{\triangleleft}, \mathcal{C}) \times_{\mathrm{Fun}(K,\mathcal{C})} \{p\}$. Dually, we will say that a right cone $\overline{p}: K^{\triangleright} \longrightarrow \mathcal{C}$ is a *colimit cone* if it is initial as an object of $\mathrm{Fun}(K^{\triangleright}, \mathcal{C}) \times_{\mathrm{Fun}(K,\mathcal{C})} \{p\}$.

We now arrive to the definition of stable ∞-categories.

2.0.10. **Definition.** Let $\mathcal{C}$ be an ∞-category and let $X \in \mathcal{C}$ be an object. We will say that X is a *zero object* of $\mathcal{C}$ if it is both initial and terminal. We will say that $\mathcal{C}$ is *pointed* if it contains a zero object. We will usually denote a 0-object by $0 \in \mathcal{C}$. We note that while there may be two different objects of $\mathcal{C}$ which are 0-objects, the full subcategory spanned by zero objects is contractible. We will say that a functor $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{D}$ between pointed ∞-categories is *reduced* if it preserves zero objects.

2.0.11. **Definition.** Let $\mathcal{C}$ be a pointed ∞-category. We will say that $\mathcal{C}$ is *stable* if the following conditions hold: An ∞-category $\mathcal{C}$ is said to be *stable* if it satisfies the following conditions

i) $\mathcal{C}$ is pointed.
ii) $\mathcal{C}$ admits both pushouts and pullbacks.
iii) A square

$$\begin{tikzcd} x \arrow[r] \arrow[d] & y \arrow[d] \\ z \arrow[r] & w \end{tikzcd}$$

in $\mathcal{C}$ is a pushout square if and only if it is a pullback square. We will refer to such squares as *exact squares*. When z (or y) are the zero object we will also refer to such squares as *exact sequences*.

2.0.12. **Definition.** Let $\mathcal{C}, \mathcal{D}$ be stable ∞-categories. We will say that a functor $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{D}$ is a *exact* if it is reduced and preserves exact squares.

2.0.13. **Example.** Let $\mathcal{A}$ be an abelian category with enough projectives/injectives. Then the various derived categories associated to $\mathcal{A}$ are stable.

2.0.14. **Example.** Recall that a *spectrum* is a sequence of spaces $X_0, X_1, ...$, together with equivalences $X_i \xrightarrow{\simeq} \Omega X_{i+1}$. The collection of spectra can be organized into an ∞-category by taking the coherent nerve of a suitable topological/Kan enriched category. It is a stable ∞-category.

Forming the cofibre of the essentially unique arrow $x \to 0$ we obtain the suspension $\Sigma_{\mathcal{C}} x$ of x in $\mathcal{C}$. Forming suspensions can then be organized into a functor

$$\Sigma_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}.$$

Dually, the fibre of the map $0 \to x$ we obtain the loop $\Omega_{\mathcal{C}} x$ of x in $\mathcal{C}$, and this can again be organized into a functor

$$\Omega_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}.$$

These constructions can be performed whenever $\mathcal{C}$ is a pointed ∞-category which admits fibres and cofibres. When $\mathcal{C}$ is stable, these operations are inverses to each other, and are in particular equivalences. In fact, this property characterized stable ∞-categories:

2.0.15. **Proposition.** *Let $\mathcal{C}$ be a pointed ∞-category. Then the following assertions are equivalent:*

i) $\mathcal{C}$ is stable.
ii) Every map in $\mathcal{C}$ admits a cofiber and the suspension functor is an equivalence.
iii) Every map in $\mathcal{C}$ admits a fiber and the loop functor is an equivalence.

The fact that the suspension and loops in a stable ∞-category are inverse equivalences has some interesting consequences:

i) The homotopy category of a stable ∞-category inherits the structure of a triangulated category.
ii) If $x, y \in \mathcal{C}$ are objects then the sequence of spaces $X_i := \mathrm{Map}_{\mathcal{C}}(x, \Sigma^i y)$ satisfy

$$\Omega X_{i+1} = \Omega \,\mathrm{Map}_{\mathcal{C}}(x, \Sigma^{i+1} y) = \mathrm{Map}_{\mathcal{C}}(x, \Omega\Sigma^{i+1} y) = \mathrm{Map}_{\mathcal{C}}(x, \Sigma^i y)$$

and hence upgrade to form a spectrum, which we denote by $\mathrm{map}_{\mathcal{C}}(x, y)$, and call it the *mapping spectrum* from x to y. One can show that this construction endows $\mathcal{C}$ with an enrichment in spectra.
iii) By construction $\mathrm{Map}_{\mathcal{C}}(x, y) = \Omega^\infty \mathrm{map}_{\mathcal{C}}(x, y)$. In particular, all the mapping spaces in $\mathcal{C}$ are infinite loop spaces, and are hence all carry an E_∞-group structure.

Quillen's definition of the algebraic K-theory space can be extended from exact/abelian categories to stable ∞-categories. In fact, the definition is in some sense simpler: if $\mathcal{C}$ is a stable ∞-category then we have no notion of injective or surjective maps (or, in some since, they are all injective and surjective) and so we can form the span ∞-category $\mathrm{Span}(\mathcal{C})$ whose objects are the objects of $\mathcal{C}$ and whose morphisms are spans

$$\begin{tikzcd} & w \arrow[dl] \arrow[dr] & \\ x & & y \end{tikzcd}$$

without requiring any conditions on f and g. The 2-simplices in $\mathrm{Span}(\mathcal{C})$ are then given by diagrams

$$\begin{tikzcd} & & u \arrow[dl] \arrow[dr] & & \\ & w \arrow[dl] \arrow[dr] & & v \arrow[dl] \arrow[dr] & \\ x & & y & & z \end{tikzcd}$$

in which the middle square is exact. More generally, if we let $\mathrm{TwAr}^{\mathrm{r}}[n]$ be the twisted arrow category of the poset $[n]$ (so that $\mathrm{TwAr}^{\mathrm{r}}[n]$ is the poset whose elements are arrows $i \leq j$ in $[n]$ and such that a unique arrow exists from (i, j) to (i', j') if $i \leq i' \leq j' \leq j$), then the n-simplices in $\mathrm{Span}(\mathcal{C})$ are the given by diagrams $\varphi\colon \mathrm{TwAr}^{\mathrm{r}}[n] \to \mathcal{C}$ satisfying the following condition: for every $i \leq j \leq k \in [n]$ the square

$$\begin{tikzcd} \varphi(i \leq k) \arrow[r] \arrow[d] & \varphi(i \leq j) \arrow[d] \\ \varphi(j \leq k) \arrow[r] & \varphi(j \leq j) \end{tikzcd}$$

is exact.

We then define the K-theory space of $\mathcal{C}$ by the formula

$$\mathcal{K}(\mathcal{C}) = \Omega|\,\mathrm{Span}(\mathcal{C})|.$$

Note that, since we don't require any surjection/inclusion condition on the maps in our spans, for every object x the span

$$\begin{tikzcd} & x \arrow[dl] \arrow[dr] & \\ 0 & & 0 \end{tikzcd}$$

is a morphism in $\mathrm{Span}(\mathcal{C})$. Forsaking any distinction beween spaces and ∞-groupoids, one can then use this to construction a natural map

$$\mathcal{C}^{\simeq} \to \Omega|\,\mathrm{Span}(\mathcal{C})| = \mathcal{K}(\mathcal{C}).$$

This formalism can be used as an alternative to Quillen's setting of exact categories. For this, let us first point out that $\mathcal{K}$ is product preserving, and so, as with exact categories, the direct sum operation endows $\mathcal{K}(\mathcal{C})$ with an E_∞-group structure (and in particular, a binary product associative and commutative up to homotopy). Second, the fundamental property of additivity extends to the setting of stable ∞-categories. In particular, if we define $\mathrm{Ex}(\mathcal{C})$ as the stable ∞-category of exact sequences $x \to y \to z$ then $\mathrm{K}(\mathrm{Ex}(\mathcal{C})) \simeq \mathrm{K}(\mathcal{C}) \times \mathrm{K}(\mathcal{C})$. Second, we can study algebraic K-theory of rings and varieties using stable ∞-categories instead of exact ones. To do so, recall that an object in the derived category of a scheme X is called *perfect* if it is affine locally equivalent to a bounded complex of finitely generated projective modules. A scheme is said to be divisorial if every perfect complex is equivalent to a bounded complex of vector bundles. For example, every quasi-projective scheme is divisorial.

2.0.16. **Theorem** (Gillet-Waldhausen)**.** *Let X be a divisorial scheme. Then*

$$\mathcal{K}(\mathrm{Vect}(X)) \simeq \mathcal{K}(\mathcal{D}^{\mathrm{p}}(X)).$$

In particular, we may construction the algebraic K-theory of X via its perfect derived ∞-category, as opposed to vector bundles.

We may view this as a higher categorical version of Quillen's resolution theorem: the passage from $\mathrm{Vect}(X)$ to $\mathcal{D}^{\mathrm{p}}(X)$ exactly relies on the fact that every perfect complex admits a finite resolution by vector bundles when X is divisorial. If X is not divisorial then we consider the "correct" algebraic K-theory space to be $\mathcal{K}(\mathcal{D}^{\mathrm{p}}(X))$, and not $\mathcal{K}(\mathrm{Vect}(X))$. This is mostly harmless (since the collection of divisorial schemes is rather large), and can be a-posteriori justified by the fact that K-theory of $\mathcal{D}^{\mathrm{p}}(X)$ gives a better behaved invariant of X. For example, it satisfies the Mayer-Vietoris property, that is, for an open covering $X = U \cup V$ with intersection $W = V \cap U$ then the square

$$\begin{tikzcd} \mathcal{K}(\mathcal{D}^{\mathrm{p}}(X)) \arrow[r] \arrow[d] & \mathcal{K}(\mathcal{D}^{\mathrm{p}}(U)) \arrow[d] \\ \mathcal{K}(\mathcal{D}^{\mathrm{p}}(V)) \arrow[r] & \mathcal{K}(\mathcal{D}^{\mathrm{p}}(W)) \end{tikzcd}$$

is cartesian, even if X is not regular (though we note that in the non-regular case the resulting long exact sequence of K-groups is not necessarily surjective on the last K_0 term). This was first proven by Thomason and Trobaugh [TT90] almost 20 years after Quillen's proved his localisation theorem for abelian categories, and they refer to this as a revolutionary result. Indeed, not having localisation statements for algebraic K-theory for non-regular varieties was considered a real block to the development of algebraic K-theory of varieties. While Thomason and Trobaugh also worked with the perfect derived category of a scheme, they did not consider it as an ∞-category (as this notion was not yet developed at the time), and used instead the formalism of categories equipped with a collection of weak equivalences (and possibly additional structure, a setting in which a lot of higher category was set in before the usage of ∞-categories became prevalent). In modern terms, the above Mayer-Vietoris property for algebraic K-theory is best viewed as a consequence of a localisation theorem for algebraic K-theory of stable ∞-categories, analogous to the localisation theorem proved by Quillen for abelian categories. Here, we replace the notion of a Serre subcategory by that of a full stable subcategory $\mathcal{A} \subseteq \mathcal{B}$ which is closed under retracts. The Serre quotient we had in abelian categories is then replaced with the Verdier quotient $\mathcal{A}/\mathcal{B}$, whose objects are the objects of $\mathcal{B}$ and whose mapping spaces are given by the colimits

$$\mathrm{Map}_{\mathcal{A}/\mathcal{B}}(x,y) = \operatorname*{colim}_{\substack{f\,:\,x'\to y\\ \mathrm{fib}(f)\in\mathcal{A}}} \mathrm{Map}_{\mathcal{B}}(x',y) = \operatorname*{colim}_{\substack{f\,:\,y\to y'\\ \mathrm{fib}(f)\in\mathcal{A}}} \mathrm{Map}_{\mathcal{B}}(x,y').$$

The corresponding *Verdier sequence*

$$\mathcal{A} \to \mathcal{B} \to \mathcal{B}/\mathcal{A}$$

is then a fibre-cofibre sequence of stable ∞-categories. The following can be viewed as a modern avatar of Waldhausen's fibration theorem. It was proven in the setting of stable ∞-categories by Hebestreit-Lachmann-Steimle.

2.0.17. **Theorem** (Localisation)**.** *For every Verdier sequence $\mathcal{A} \to \mathcal{B} \to \mathcal{B}/\mathcal{A}$ of stable ∞-categories the sequence*

$$\mathcal{K}(\mathcal{A}) \to \mathcal{K}(\mathcal{B}) \to \mathcal{K}(\mathcal{B}/\mathcal{A})$$

is a fibre sequence of spaces.

Another advantage of the point of view of stable ∞-categories is that it enables one to give the following universal characterization of algebraic K-theory:

2.0.18. **Theorem** (Blumberg-Gepner-Tabuada)**.** *As a functor on stable ∞-categories, algebraic K-theory is the universal group-like additive functor receiving a natural transformation from the core groupoid functor.*

Finally, let us also explain how to view algebraic K-theory of a stable ∞-category not just as a space but as a spectrum. Recall that by construction $\mathcal{K}(\mathcal{C}) = \Omega|\mathrm{Span}(\mathcal{C})|$ is a loop of something, namely, the loop of the realization of the Q-construction. Recall also that the geometric realization $|\mathrm{Span}(\mathcal{C})|$ is equivalent to the geometric realization of the simplicial space $|\mathrm{Q}_\bullet(\mathcal{C})^{\simeq}|$ where $\mathrm{Q}_m(\mathcal{C}) := \mathrm{Fun}(\mathrm{TwAr}^{\mathrm{r}}[m], \mathcal{C})$. Now since each $\mathrm{Q}_m(\mathcal{C})$ is also a stable ∞-category we can iterate this construction, yielding an n-fold simplicial space $\mathrm{Q}^{(n)}_{\bullet,\dots,\bullet}(\mathcal{C})^{\simeq}$, where $\mathrm{Q}^{(n)}_{m_1,\dots,m_n}(\mathcal{C}) = \mathrm{Q}_{m_1}\mathrm{Q}_{m_2}\dots\mathrm{Q}_{m_n}(\mathcal{C})$. One can then show that $|\mathrm{Q}^{(n)}(\mathcal{C})| = \Omega|\mathrm{Q}^{(n+1)}|$, so that each $\mathrm{Q}^{(n+1)}(\mathcal{C})$ is a delooping of $\mathrm{Q}^{(n)}(\mathcal{C})$. The sequence of spaces

$$\mathcal{K}(\mathcal{C}), |\mathrm{Q}_\bullet(\mathcal{C})|, |\mathrm{Q}^{(2)}_{\bullet,\bullet}(\mathcal{C})|, \dots$$

together with the structure maps $|\mathrm{Q}^{(n)}(\mathcal{C})| \xrightarrow{\simeq} \Omega|\mathrm{Q}^{(n+1)}|$ determine a refinement of $\mathcal{K}$ to K-theory *spectrum* $\mathrm{K}(\mathcal{C})$. One can then further show that for each n the space $|\mathrm{Q}^{(n)}(\mathcal{C})|$ is $(n-1)$-connected, and so the K-theory spectrum is *connective*. May's recognition principle then implies that this spectral refinement is in fact equivalent to the data of an E_∞-group structure we have discussed above. The universal property of Blumberg-Gepner-Tabuada can then be formulated also for the K-spectrum, namely, the functor

$$\mathrm{K} : \mathrm{Cat}^{\mathrm{ex}}_\infty \to \mathcal{S}p$$

is the initial additive functor equipped with a natural transformation $(-)^{\simeq} \Rightarrow \mathrm{K}(-)$.

3. Trace methods

Having established algebraic K-theory as something that can be regarded as a cohomology theory for schemes, we would like to better understand what type of information it captures. For this, recall that a basic type of cohomological invariants of schemes encountered relatively early on are those coming from cohomology of quasi-coherent sheaves. For example, on a smooth variety we have the Hodge cohomology groups $H^i(X, \Omega^j_X)$, and regardless of smoothness we can always consider the cohomologies $H^i(X, \mathcal{O}_X)$ of the structure sheaf. These cohomological invariants have the property that if X is a variety over a field k then the invariants are k-vector spaces, and for a scheme over a ring R these invariants are R-modules.

Let us call such invariants “linear”. We can also formulate this in terms of characteristics: for schemes in characteristic p these invariants are purely p-local, and have trivial q-local information for $q \neq p$.

Algebraic K-theory is not such an invariant, as can be seen from the case of algebraically closed fields, see Example 1.0.2. We may however ask what type of linear information is seen by algebraic K-theory. For example, we can then ask if there are interesting linear invariants to which algebraic K-theory maps to. Such examples are given by classical trace maps.

3.0.1. **Construction.** Let R be a commutative ring. Then every projective R-module P is a dualizable object in $\mathrm{Mod}(R)$, with dual being $P^\vee := \mathrm{Hom}_R(P, R)$. In particular, $P \otimes_R P^\vee \cong \mathrm{Hom}_R(P, P)$ and we have evaluation and coevaluation maps

$$R \to P \otimes_R P^\vee \to R$$

whose composite is the trace $\mathrm{tr}(V) \in \mathrm{End}_R(R) = R$. Now traces are additive, so that $\mathrm{tr}(P \oplus Q) = \mathrm{tr}(P) + \mathrm{tr}(Q)$, and so descent to a group homomorphism

$$\mathrm{K}_0(R) \to R.$$

This construction can also be adapted to the case where R is not commutative, in which case it lands in the quotient abelian group $R/[R, R]$ of R modulue the subgroup generated by the commutators $ab - ba$.

Recall that for a field k and an associative k-algebra R the Hochschild homology of R over k is given by

$$\mathrm{HH}_n(R/k) := \mathrm{Tor}_n^{R\otimes_k R^{\mathrm{op}}}(R, R) = \mathrm{H}_n(R \otimes^{\mathrm{L}}_{R\otimes_k R^{\mathrm{op}}} R).$$

In particular $\mathrm{HH}_0(R/k) = R/[R, R]$ (regardless of k). For associative algebras over a commutative ring k the above trace construction was then extended by Dennis to a map

$$\mathrm{K}_n(R) \to \mathrm{HH}_n(R/k)$$

from the n’th K-group to the n’th Hochschild homology group. In the setting of k-varieties, one can similarly obtain trace maps

$$\mathrm{K}_n(X) \to \mathrm{HH}_n(X/k) = \hat{\mathrm{H}}^{-n}(\mathrm{L}\,\Delta^*\Delta_*\mathcal{O}_X)$$

where $\Delta\colon X \to X \times_k X$ is the diagonal map and on the right hand side we have the n’th hyper-cohomology of the complex $\mathrm{L}\,\Delta^*\Delta_*\mathcal{O}_X$, where $\mathrm{L}\,\Delta^*$ is the total derived functor of Δ^*. For example, by the Hochschild-Kostant-Rosenberg theorem if X is a smooth and proper k-variety of dimension d then we have $\mathrm{L}\,\Delta^*\Delta_*\mathcal{O}_X = \oplus_{i=0}^d \Omega^i_X[i]$ and the Hochschild homology groups

$$\mathrm{HH}_n(X) = \bigoplus_{i=0}^{n} \mathrm{H}^{i-n}(X, \Omega^i)$$

are direct sums of Hodge cohomology groups. At any rate, the Hochshild homology groups are a linear invariant in the sense above sending k-varieties to k-vector spaces. The Dennis trace map

$$\mathrm{K}_n(X) \to \mathrm{HH}_n(X/k)$$

is then a non-trivial map to a linear invariant capturing at least some of the “on characteristic” information in $\mathrm{K}(X)$.

It is natural to ask if this trace map is in some sense optimal, or universal, in terms of capturing on characteristic information. There is however a natural refinement of the Dennis trace, discovered by Bökstedt, where we replace the Hochschild homology on the right hand side with *topological Hochschild homology*, or THH. In addition, instead of having a trace map defined on the level of each individual K-group, there is a single map on the level of K-theory *spectra*.

Now in the setting of stable ∞-categories one can define topological Hochschild homology $\mathrm{THH}(\mathcal{C})$ by the formula

$$\mathrm{THH}(\mathcal{C}) := \operatorname*{colim}_{[x\to y]\in \mathrm{TwAr}^{\mathrm{r}}(\mathcal{C})} \mathrm{map}(y, x) \in \mathrm{Sp}$$

where map denotes mapping spectra, and $\mathrm{TwAr}^{\mathrm{r}}(\mathcal{C})$ is the twisted arrow category of $\mathcal{C}$ whose objects are the arrows $x \to y$ and whose morphisms $[x \to y] \mapsto [x' \to y']$ are given by commutative diagrams of the form

$$\begin{array}{ccc} x & \longrightarrow & y \\ \downarrow & & \uparrow \\ x' & \longrightarrow & y'. \end{array}$$

3.0.2. **Example.** Given an associative ring R, the topological Hochschild homology of $\mathcal{D}^{\mathrm{p}}(R)$ is given by a formula similar to $\mathrm{HH}(R)$, only instead of tensor product over some base field k one takes tensor products over the sphere spectrum. More precisely, one may embed abelian groups in spectra via the Eilenberg-Maclane functor

$$\mathrm{H}\colon \mathcal{A}b \to \mathcal{S}p.$$

This functor is lax symmetric monoidal, and hence sends rings to ring spectra. The topological Hochschild homology of R is then given by

$$\mathrm{THH}(R) := \mathrm{THH}(\mathcal{D}^{\mathrm{p}}(R)) = \mathrm{H}(R) \otimes_{\mathrm{H}(R)\otimes_{\mathbb{S}}\mathrm{H}(R^{\mathrm{op}})} \mathrm{H}(R).$$

3.0.3. **Example.** Bökstedt famously calculated the topological Hochschild homology of $\mathbb{F}_p$, and showed that

$$\pi_*\mathrm{THH}(\mathbb{F}_p) = \mathbb{F}_p[\mu] \qquad |\mu| = 2.$$

Hesselholt and Madsen generalized this to an arbitrary perfect field k of characteristic p, and also showed that for any smooth k-algebra A (over such a perfect field k) one has

$$\pi_*\mathrm{THH}(A) = \Omega^*_{A/k} \otimes_k k[\mu].$$

3.0.4. **Remark.** If R is commutative then we can also identify $\mathrm{THH}(R)$ with the colimit in commutative (more precisely, E_∞-) ring spectra of the constant S^1-indexed diagram taking the value $\mathrm{H}(R)$. Indeed, tensor products are coproducts of E_∞-rings and relative tensor products are pushouts. In the category of E_∞-ring spectra we hence have a pushout square

$$\begin{array}{ccc} \mathrm{H}(R) \amalg \mathrm{H}(R) & \longrightarrow & \mathrm{H}(R) \\ \downarrow & & \downarrow \\ \mathrm{H}(R) & \longrightarrow & \mathrm{THH}(R) \end{array}$$

where the coproduct is also to be understood in E_∞-ring spectra, and since $S^1 = * \amalg_{*\amalg *} *$ in $\mathcal{S}$ this exhibits $\mathrm{THH}(R)$ as the colimit of the constant S^1-indexed diagram on $\mathrm{H}(R)$.

To construct the trace map

$$\mathrm{K}(\mathcal{C}) \to \mathrm{THH}(\mathcal{C})$$

one can use the universal property of K-theory by Blumberg-Gepner-Tabuada. In particular, one first shows that THH is an additive invariant on stable ∞-categories. For a stable ∞-category $\mathcal{C}$ one then considers the composite map

$$\begin{aligned} \mathcal{C}^{\simeq} &\to \mathrm{Map}(S^1, \mathcal{C}^{\simeq}) \\ &\to \operatorname*{colim}_{[x\to y]\in \mathrm{TwAr}^{\mathrm{r}}(\mathcal{C})} \mathrm{Map}(y, x) \\ &\to \Omega^\infty \operatorname*{colim}_{[x\to y]\in \mathrm{TwAr}^{\mathrm{r}}(\mathcal{C})} \mathrm{map}(y, x) \\ &= \Omega^\infty \mathrm{THH}(\mathcal{C}). \end{aligned}$$

By the universal property of the K-theory spectrum this map determines a natural transformation

$$\mathrm{K} \Rightarrow \mathrm{THH}.$$

Now if R is commutative then $\mathrm{THH}(R)$ is an E_∞-ring spectrum equipped with a map from $\mathrm{H}(R)$, and so all its homotopy groups are R-modules. In particular, THH is still a linear invariant of commutative rings. More generally, THH sends R-linear categories to $\mathrm{H}(R)$-module spectra, so that, if X is a scheme over R then the homotopy groups of $\mathrm{THH}(X)$ are R-modules. We may hence revisit the question of universality of the trace map $\mathrm{K} \Rightarrow \mathrm{THH}$ in terms of capturing linear information.

For this, let us try to be more precise about what linear really means. To incorporate example like THH let us consider that our invariant takes values in spectra. One manner to produce invariants exhibiting the linear property above is to consider the category if pairs (X, F) where X is a scheme and $F \in \mathcal{D}^{qc}(X)$ is an object in the quasi-coherent derived ∞-category. Such a category could be denoted as $\int_{X \in \mathrm{Sch}} \mathcal{D}^{qc}(X)$, being the Grothendieck construction of the functor $X \mapsto \mathcal{D}^{qc}(X)$. Now suppose that T is a spectrum-valued invariant of such pairs such that for every X, the restriction of T to quasi-coherent sheaves over X is exact. We can then view T as an invariant of schemes by setting $T(X) := T(X, \mathcal{O}_X)$. For example, we can imagine that $T(X, F) = R\Gamma(X, F)$ is the total derived functor of global sections. Now if X is a scheme over R then $\mathcal{D}^{qc}(X)$ is an R-linear stable ∞-category and so $\mathrm{H}(R)$ acts on every object in $\mathcal{D}^{qc}(X)$. In particular, by functoriality and exactness, $T(X, \mathcal{O}_X)$ would be an $\mathrm{H}(R)$-module. Let us hence take the above as a formal definition of what being linear means. This definition applies to THH. In fact, $\mathrm{THH}(\mathcal{C})$ extends naturally to allow coefficients in M a $\mathcal{C}$-bimodule, that is, a bi-exact functor $\mathcal{C}^{op} \times \mathcal{C} \to \mathrm{Sp}$. The version of THH with coefficients is then given by

$$\mathrm{THH}(\mathcal{C}, M) = \operatorname*{colim}_{[x \to y] \in \mathrm{TwAr}^r(\mathcal{C})} M(y, x).$$

In the case of $\mathcal{C} = \mathcal{D}^p(X)$ for a scheme X, we may, if we wish, restrict attention to $\mathcal{D}^p(X)$-bimodules of the form $M_F(P, Q) := \mathrm{map}(P, F \otimes Q)$ for some $F \in \mathcal{D}^{qc}(X)$. In particular, the THH as a functor on schemes is indeed linear in the precise sense we proposed above.

Now the algebraic K-theory functor on schemes does not admit a-priori a linear extension to coefficients. It does admit another extension, coming from the fact that K-theory is fundamentally a functor of categories rather than schemes. In particular, we may consider the category $\int_{X \in \mathrm{Sch}} \mathrm{Mod}_{\mathcal{D}^p(X)}(\mathrm{Cat}^{ex}_\infty)_{/\mathcal{D}^p(X)}$ of pairs $(X, \mathcal{C})$ where X is a scheme and $\mathcal{C}$ is a $\mathcal{D}^p(X)$-linear stable ∞-category equipped with a $\mathcal{D}^p(X)$-linear functor $\mathcal{C} \to \mathcal{D}^p(X)$. We may view $\mathcal{C}$ as a sheaf of categories on X (equipped with a map to the "structure sheaf"). We can then define $\mathrm{K}(X, \mathcal{C}) := \mathrm{K}(\mathcal{C})$, and view this as a sort of extension of algebraic K-theory to admit coefficients. Now consider the sequence of functors

$$\mathrm{Sch} \to \int_{X \in \mathrm{Sch}} \mathrm{Mod}_{\mathcal{D}^p(X)}(\mathrm{Cat}^{ex}_\infty)_{/\mathcal{D}^p(X)} \xrightarrow{\Phi} \int_{X \in \mathrm{Sch}} \mathcal{D}^{qc}(X)$$

where the first functor is given by $X \mapsto (X, \mathcal{D}^p(X))$ and $\Phi(X, f : \mathcal{C} \to \mathcal{D}^p(X)) = (X, F)$ where F is the quasi-coherent sheaf

$$F := \operatorname*{colim}_{(x, f(x) \to \mathcal{O}_X) \in \mathcal{C}_{/\mathcal{O}_X}} f(x).$$

Then the composite of the above two functors gives $X \mapsto (X, \mathcal{O}_X)$. We may then ask the following question: what is the universal extension of K from a functor defined on $\int_{X \in \mathrm{Sch}} \mathrm{Mod}_{\mathcal{D}^p(X)}(\mathrm{Cat}^{ex}_\infty)_{/\mathcal{D}^p(X)}$ to a functor defined on $\int_{X \in \mathrm{Sch}} \mathcal{D}^{qc}(X)$? More precisely, what is the universal functor on $\int_{X \in \mathrm{Sch}} \mathcal{D}^{qc}(X)$ whose restriction along F admits a map from algebraic K-theory? The answer to this more precise question is that this would correspond to the left Kan extension of K along Φ. But recall that to be linear we required not just a functor on $\int_{X \in \mathrm{Sch}} \mathcal{D}^{qc}(X)$, but a functor which is exact in the $\mathcal{D}^{qc}(X)$ direction. The universal extension we are after is hence the fibre-wise exact approximation of the left Kan extension of K along Φ.

How might such a thing be computed? Well, one can observe that the functor Φ admits a right adjoint

$$\Psi : \int_{X \in \mathrm{Sch}} \mathcal{D}^{qc}(X) \to \int_{X \in \mathrm{Sch}} \mathrm{Mod}_{\mathcal{D}^p(X)}(\mathrm{Cat}^{ex}_\infty)_{/\mathcal{D}^p(X)}$$

sending (X, F) to the pair $(X, \mathrm{Lace}(\mathcal{D}^p(X), F))$ where $\mathrm{Lace}(\mathcal{D}^p(X), F)$ is the ∞-category of pairs (P, α) where $P \in \mathcal{D}^p(X)$ is a perfect complex and $\alpha : P \to F \otimes P$ is a map (or an *F-lacing of P*). We can view this as an F-twisted version of the category of objects equipped with endomorphisms. The left Kan extension $\Phi_! \mathrm{K}$ is then given by pre-composing with Ψ, namely, we obtain the functor

$$\mathrm{K}^{\mathrm{lace}}(X, F) := \mathrm{K}(\mathrm{Lace}(\mathcal{D}^p(X), F)).$$

Now this is just the left Kan extension, but the result is not yet exact in F. We now take its exact approximation in F. This procedure consists in two steps. The first step is to make $\mathrm{K}^{\mathrm{lace}}$ reduced, that is, zero preserving, in F. The universal way of doing so is to replace $\mathrm{K}^{\mathrm{lace}}$ by

$$\mathrm{K}^{\mathrm{cyc}}(X, F) = \mathrm{fib}[\mathrm{K}^{\mathrm{lace}}(X, F) \to \mathrm{K}(X)].$$

Next, we universally turn the result into an exact functor by taking the sequential colimit

$$\partial_F \mathrm{K}^{\mathrm{lace}}(X, F) := \mathrm{colim}[\mathrm{K}^{\mathrm{cyc}}(X, F) \to \Omega \mathrm{K}^{\mathrm{cyc}}(X, \Sigma F) \to \Omega^2 \mathrm{K}^{\mathrm{cyc}}(X, \Sigma^2 F) \to ....].$$

The following theorem was proven (in a slightly different formulation) for rings by Dundas-McCarthy, and for general stable ∞-categories in [HNS24]:

3.0.5. **Theorem.** *There is a natural equivalence*

$$\partial_F \mathrm{K}^{\mathrm{cyc}}(X, F) \simeq \mathrm{THH}(X, F) := \mathrm{THH}(\mathcal{D}^{\mathrm{p}}(X), M_F).$$

3.0.6. **Remark.**

i) In the above theorem, one can show that both sides are colimit preserving functors of F. It is hence sufficient, by purely formal reasons, to identify them as functors on connective quasi-coherent sheaves $F \in \mathcal{D}^{\mathrm{qc}}_{\geq 0}(X)$, or even on m-connective quasi-coherent sheaves for some fixed m.
ii) Suppose that $X = \mathrm{spec}(R)$ for a ring R. Then one can show that $\mathrm{Lace}(R, M)$ is generated under finite colimits and retracts by the object $(R, 0)$, and so

$$\mathrm{Lace}(R, M) \simeq \mathrm{Mod}^{\omega}(\mathrm{End}(R, 0)) = \mathrm{Mod}^{\omega}(M[-1] \oplus R)$$

where $M[-1] \oplus R$ is the split square-zero extension of R by the connective R-module complex $M[-1]$ (viewed as a dg-ring, or a ring spectrum). This was the setting in which Dundas-McCarthy originally proved that theorem, namely, derivation with respect to taking square-zero extensions, and so their result carries a shift corresponding to the gap between M and $M[-1]$ above.

We may view this theorem as saying that the trace map $\mathrm{K}(X) \to \mathrm{THH}(X)$ is the universal approximation of K by an on characteristic functor. Does this mean that $\mathrm{THH}(X)$ sees all on characteristic information seen by K? Not exactly. For example, if we consider the fibre $\mathrm{K}' = \mathrm{fib}[\mathrm{K} \to \mathrm{THH}]$ then there could be non-trivial linear invariants to which K' maps to, the universal of which could be obtained by a derivation procedure as above. It can be helpful to think of the following analogy. Instead of invariants on schemes, consider simply abelian groups, and replace being linear by being p-torsion (for some fixed p). Given an abelian group M, the universal p-torsion group to which M maps to is M/p. But the kernel of $M \to M/p$ might still contain p-torsion, since M corresponding to non-trivial p^2-torsion in M. The map $M \to M/p$ hence does not see all p-primary torsion information in M. What we could do, in this case, is to consider the p-completion $M_p := \lim_i M/p^i$. Then the map $M \to M_p$ would see all p-primary torsion, and all p-local information. Furthermore, both the kernel and cokernel of the map $M \to M_p$ are $\mathbb{Z}[p^{-1}]$-modules, so the part of M not seen by M_p is supported away from p.

Going back to algebraic K-theory, can we in a similar manner replace the trace map $\mathrm{K}(X) \to \mathrm{THH}(X)$ by a map acting as "completion along the characteristic". For this, one needs to pass from the trace map to the cyclotomic trace map. To explain this point, one first notes that THH of any ring, scheme or category carries a canonical circle action. In the case of commutative rings, this can be seen via the equivalence of Remark 3.0.4, but this is also true for non-commutative rings, and can be proven, for example, by extending the bar construction computing the relevant relative tensor product to a cyclic object. Furthermore, the trace map

$$\mathrm{K}(X) \Rightarrow \mathrm{THH}(X)$$

is S^1-equivariant, where we view S^1 as acting trivially on K. Secondly, the circle action on $\mathrm{THH}(X)$ refines to a *cyclotomic structure*. This means, in particular, that $\mathrm{THH}(X)$ carries S^1-equivariant Frobenius maps

$$F_p : \mathrm{THH}(X) \to \mathrm{THH}(X)^{\mathrm{t}\mu_p}$$

where $\mu_p \subseteq S^1$ is the p-torsion subgroup, which is just a cyclic group of order p, and $(-)^{\mathrm{t}\mu_p} = \mathrm{cof}[(-)_{\mathrm{h}\mu_p} \to (-)^{\mathrm{h}\mu_p}]$ is the *Tate construction*. Here, the right hand side carries an induced action of S^1/μ_p which we identify with S^1 via the exact sequence $\mu_p \to S^1 \xrightarrow{p} S^1$. These are again compatible with the trace map in the since that we have commutative diagrams

$$\begin{tikzcd}
\mathrm{K}(X) \arrow[r] \arrow[d] & \mathrm{THH}(X) \arrow[dd, "F_p"] \\
\mathrm{K}^{\mathrm{h}\mu_p}(X) \arrow[d] & \\
\mathrm{K}^{\mathrm{t}\mu_p}(X) \arrow[r] & \mathrm{THH}(X)^{\mathrm{t}\mu_p}
\end{tikzcd}$$

where the map $\mathrm{K}(X) \Rightarrow \mathrm{K}^{\mathrm{h}\mu_p}(X)$ arises from the fact that we are working with the trivial circle action on $\mathrm{K}(X)$. This implies that the trace map refines to a map

$$\mathrm{K}(X) \Rightarrow \mathrm{TC}(X) := \mathrm{Eq}\Big[\mathrm{THH}(X)^{\mathrm{h}S^1} \rightrightarrows \prod_p (\mathrm{THH}(X)^{\mathrm{t}\mu_p})^{\mathrm{h}(S^1/\mu_p)}\Big],$$

where the two maps in the equalizer are the map induced on $S^1 = S^1/\mu_p$ homotopy fixed points by the Frobenius $\mathrm{THH}(X) \Rightarrow \mathrm{THH}(X)^{\mathrm{t}\mu_p}$ and the composite

$$\mathrm{THH}^{\mathrm{h}S^1} = (\mathrm{THH}(X)^{\mathrm{h}\mu_p})^{\mathrm{h}S^1/\mu_p} \to (\mathrm{THH}(X)^{\mathrm{t}\mu_p})^{\mathrm{h}(S^1/\mu_p)}.$$

The functor $\mathrm{TC}(X)$ is then called *topological cyclic homology* and the resulting map the *cyclotomic trace map*.

3.0.7. **Remark.** For quasi-compact quasi-separated schemes X, $\mathrm{THH}(X)$ is a bounded below spectrum, and it is in this generality that TC can be defined by the above equalizer formula. In general, the formula for TC is more complicated.

The fibre $\mathrm{K}^{\mathrm{inf}}(X) := \mathrm{fib}[\mathrm{K}(X) \to \mathrm{TC}(X)]$ of the cyclotomic trace map is called infinitesimal K-theory. We claim that $\mathrm{K}^{\mathrm{inf}}$ has no maps to linear invariants. This is due to the celebrated Dundas-Goodwillie-McCarthy theorem. In what follows, we say that a ring homomorphism $R \to S$ is a nilpotent extension if it is surjective and its ideal I satisfies $I^n = 0$ for some n. This concept can also be extended to maps of dg-rings and ring spectra.

3.0.8. **Theorem** (Dundas-Goodwillie-McCarthy)**.** *Let $R \to S$ be a nilpotent extension of rings (or ring spectra). Then the square*

$$\begin{tikzcd} \mathrm{K}(R) \arrow[r] \arrow[d] & \mathrm{TC}(R) \arrow[d] \\ \mathrm{K}(S) \arrow[r] & \mathrm{TC}(S) \end{tikzcd}$$

is cartesian. In particular $\mathrm{K}^{\mathrm{inf}}$ *is invariant under nilpotent extensions.*

3.0.9. **Remark.** In their work [CMM21], Clausen-Mathew-Morrow show that the conclusion of the Dundas-Goodwillie-McCarthy theorem also holds if $R \to S$ is surjective henselien map of commutative rings.

3.0.10. **Corollary.** *Let R be a commutative ring and $M \in \mathcal{D}(R)$ a 1-connective R-complex. Then the square*

$$\begin{tikzcd} \mathrm{K}(R) \arrow[r] \arrow[d] & \mathrm{TC}(R) \arrow[d] \\ \mathrm{K}(\mathrm{Lace}(R, M)) \arrow[r] & \mathrm{TC}(\mathrm{Lace}(R, M)) \end{tikzcd}$$

is cartesian. In particular, $\mathrm{fib}[\mathrm{K}^{\mathrm{inf}}(R) \to \mathrm{K}^{\mathrm{inf}}(\mathrm{Lace}(R, M))] = 0$ *and so* $\partial_M \mathrm{K}^{\mathrm{inf}}(R) = 0$.

This theorem was extended to divisorial schemes in the thesis of Victor Sauiner:

3.0.11. **Proposition.** *Let X be a divisorial scheme and $F \in \mathcal{D}^{\mathrm{qc}}(X)$ a 1-connective quasi-coherent sheaf. Then the square*

$$\begin{tikzcd} \mathrm{K}(X) \arrow[r] \arrow[d] & \mathrm{TC}(X) \arrow[d] \\ \mathrm{K}(\mathrm{Lace}(X, F)) \arrow[r] & \mathrm{TC}(\mathrm{Lace}(X, F)) \end{tikzcd}$$

is cartesian. In particular, $\partial_M \mathrm{K}^{\mathrm{inf}}(R) = 0$.

Victor also showed that if X is not divisorial but still quasi-compact and separated, then there exists an n such that the conclusion of the last proposition remains true if F is assumed n-connective. In particular, the conclusion $\partial_M \mathrm{K}^{\mathrm{inf}}(R) = 0$ remains true for all quasi-compact separated schemes (where we recall that we work under the convention that $\mathrm{K}(X)$ means $\mathrm{K}(\mathcal{D}^{\mathrm{p}}(X))$ by definition, and similarly for TC).

To conclude, up to restricting attention to some nice class of schemes, we have that $\mathrm{K}^{\mathrm{inf}}$ has zero derivatives, and so maps trivially to any on characteristic invariant. We may hence then say that $\mathrm{K}^{\mathrm{inf}}$ is supported away from the characteristic. The exact sequence

$$\mathrm{K}^{\mathrm{inf}} \Rightarrow \mathrm{K} \Rightarrow \mathrm{TC}$$

can hence be seen as a way to break the information in K-theory to information that is supported along the characteristic and a part that is supported away from it. This point of view suggests that, when evaluating these functors on a scheme over $\mathbb{F}_p$, $\mathrm{TC}(X)$ should capture the p-adic information in $\mathrm{K}(X)$. Note that $\mathrm{TC}(X)$ is indeed p-complete for qcqs schemes over $\mathbb{F}_p$ and so we might expect that TC of such schemes might be reasonably close to the p-completion $\mathrm{K}(X)$. Summarizing work of several authors, we may extract the following statements:

3.0.12. **Proposition** (Geisser, Hesselholt, Madsen, Levin, Clausen, Mathew, Morrow)**.** *Let X be a quasi-compact, quasi-separated scheme.*

i) The restriction of TC *to the small étale site of X is a hypercomplete étale sheaf.*

ii) Let k be a perfect field of characteristic p and X a finite type k-scheme of Krull dimension d (or, more generally, a k-scheme such that the residue fields of all points are finitely generated over k with transcendence degree $\leq d$). Then $\mathrm{TC}(X)$ *is d-truncated,* $\mathrm{K}(X)$ *is p-adically d-truncated and the map*

$$\mathrm{K}(X) \to \mathrm{TC}(X)$$

is a p-adic equivalence in degrees $\geq d$. In addition, in this case the étale sheafification K^{et} *of* K *over X is hypercomplete and the map* $\mathrm{K}^{\mathrm{et}}(X) \to \mathrm{TC}(X)$ *is a p-completion.*

iii) Let k be a perfect field of characteristic p, $W(k)$ the ring of Witt vectors over k and X a proper $W(k)$-scheme of Krull dimension d. If X is a finite type and proper $\mathbb{Z}_p$-scheme of Krull dimension d then the map

$$\mathrm{K}(X) \to \mathrm{TC}(X)$$

is a p-adic equivalence in degrees $\geq d$ and the map $\mathrm{K}^{\mathrm{et}}(X) \to \mathrm{TC}(X)$ *is a p-completion.*

3.0.13. **Example.** Let $\bar{\mathbb{F}}_p$ be the algebraic closure of $\mathbb{F}_p$. Then $\mathrm{K}_0(\bar{\mathbb{F}}_p) = \mathbb{Z}$ and $\mathrm{K}_i(\bar{\mathbb{F}}_p)$ is uniquely p-divisible for $i > 0$. By the above proposition we have that $\mathrm{TC}(k) \simeq \mathrm{K}(\bar{\mathbb{F}}_p)_{\hat{p}}$ is $\mathbb{Z}_p$ concentrated in degree 0. If we now look at $\mathbb{F}_p$ then $\mathrm{K}(\mathbb{F}_p)_{\hat{p}} = \mathrm{K}(\bar{\mathbb{F}}_p)_{\hat{p}}$ is $\mathbb{Z}_p$ concentrated in degree 0 by Quillen's computations and $\mathrm{TC}(\mathbb{F}_p) = \mathbb{Z}_p^{\mathrm{h}\hat{\mathbb{Z}}}$ by étale descent. One can then verify that the Galois group of $\bar{\mathbb{F}}_p$ acts trivially on $\mathrm{TC}_0(X)$, and so $\mathrm{TC}(\mathbb{F}_p) = \mathbb{Z}_p \oplus \mathbb{Z}_p[-1]$.

3.0.14. **Example.** Let F be a finitely generated field over a perfect field k of charactersitic p, and let L be the separable closure of F. By work of illusie the map

$$L^*/(L^*)^p \to \Omega^1_L \qquad u \mapsto \frac{du}{u}$$

fits in a short exact sequence of abelian groups

$$0 \to L^*/(L^*)^p \to \Omega^1_L \xrightarrow{1-C^{-1}} \Omega^1_L/d(L) \to 0$$

where $C^{-1} : \Omega^1_L \to \Omega^1_L/d(L)$ is the inverse Cartier operator given by

$$vdu \mapsto [v^p u^{p-1} du].$$

More generally, by work of Geisser and Levine, based on work of Bloch-Kato and Gabber, for every $i \geq 1$ we have a short exact exact sequence

$$0 \to \mathrm{K}_i(L, \mathbb{Z}/p) \to \Omega^n_L \xrightarrow{1-C^{-1}} \Omega^n_L/d\Omega^{n-1}_L \to 0$$

where $C^{-1} : \Omega^i_L \to \Omega^i_L/d(L)$ given by the formula

$$vdu_1 \wedge ... \wedge du_i \mapsto [v^p u_1^{p-1} ... u_i^{p-1} du_1 \wedge ... \wedge du_i].$$

Now the mod p cyclotomic map $\mathrm{K}_i(L, \mathbb{Z}/p) \to \mathrm{TC}_i(L, \mathbb{Z}/p)$ is an isomorphism. If we now replace L by F, then the sequence

$$0 \to \mathrm{K}_i(F, \mathbb{Z}/p) \to \Omega^i_F \xrightarrow{1-C^{-1}} \Omega^i_F/d\Omega^{i-1}_F$$

is still exact, but the last map is no longer surjective. On can then show that

$$H^i_{\text{et}}(L, \Omega^i_L) = \begin{cases} \Omega^i_F & i = 0 \\ 0 & i > 0 \end{cases} \quad \text{and} \quad H^i_{\text{et}}(L, \Omega^i_L/d\Omega^{i-1}_L) = \begin{cases} \Omega^i_F/d\Omega^{i-1}_F & i = 0 \\ 0 & i > 0 \end{cases}$$

which means that

$$H^0_{\text{et}}(F, \mathrm{K}(L, \mathbb{Z}/p)) = \mathrm{K}_i(F, \mathbb{Z}/p) = \ker[\Omega^i_F \xrightarrow{1-C^{-1}} \Omega^i_F/d\Omega^{i-1}_F]$$

and

$$H^1_{\text{et}}(F, \mathrm{K}(L, \mathbb{Z}/p)) = \operatorname{coker}[\Omega^i_F \xrightarrow{1-C^{-1}} \Omega^i_F/d\Omega^{i-1}_F],$$

whereas all higher étale (or Galois) cohomology groups vanish. By étale descent one then concludes that

$$\mathrm{TC}_i(F, \mathbb{Z}/p) = \mathrm{K}_i(F, \mathbb{Z}/p) \oplus \operatorname{coker}[\Omega^{i+1}_F \xrightarrow{1-C^{-1}} \Omega^{i+1}_F/d\Omega^i_F].$$

If F has transcendence degree d over k then Ω^i_F vanishes for $i > d$ and hence both $\mathrm{TC}_i(F, \mathbb{Z}/p)$ and $\mathrm{K}_i(F, \mathbb{Z}/p)$ vanish for $i > d$.

In addition to being a major tool in understanding algebraic K-theory of schemes in positive and mixed characteristic, trace methods also have many other applications, among which are

i) Gabber rigidity: for a henselian pair (R, I) and m invertible on R, the map $\mathrm{K}(R) \to \mathrm{K}(R/I)$ is a mod m equivalence.
ii) In studying chromatic properties of algebraic K-theory, the purity property (Land-Mathew-Meier-Tamme) and descent (Clausen-Mathew-Naumann-Noel) are results which make use of trace methods in several key steps. Both these properties play a role in the proof of the redshift conjecture for E_∞-rings by Burkland-Schlank-Yuan.
iii) The recent disproof of the telescope conjecture by Burklund-Hahn-Levy-Schlank also uses trace methods, at least at one key point in the argument.

4. Hermitian K-theory and L-theory

Hermitian K-theory is a variant of K-theory where instead finitely generated projective modules we consider such modules equipped with non-degenerate quadratic forms. If algebraic K-theory can be viewed as a cohomology theory for schemes analogous to complex K-theory on topological spaces, hermitian K-theory can be considered as the analogue of real K-theory. It is hence, on the one hand, closely related, but on the other hand is sensitive to slightly different phenomenon, especially when 2 is not invertible.

Historically, different aspects of hermitian K-theory developed mostly independently in two different mathematical domains. For commutative rings, one classically defines the quadratic Grothendieck-Witt group $\mathrm{GW}^{\mathrm{q}}_0(R)$ as the group completion of the monoid of isomorphism classes $[P, q]$ of finitely generated projective modules P equipped with a non-degenerate quadratic form, that is, a form for which the induced map $q_\sharp : P \to P^\vee$ is an isomorphism. An *isotropic sub-module* inside such a (P, q) is by definition a projective sub-module $L \subseteq P$ such that $q|_L = 0$ and such that $L^\perp$ and $L^\perp/L$ are projective, where $L^\perp \subseteq P$ is the orthogonal complement of L with respect to the symmetric bilinear form $b(x, y) = q(x+y)-q(x)-q(y)$ associated to q, and we note that $L^\perp$ contains L by the assumption that $q|_L = 0$. The quotient $L^\perp/L$ is then called the associated homology, and the restriction of q to $L^\perp$ descends to a non-degenerate form q_{hom} on $L^\perp/L$. An isotropic sub-module is said to be a *Lagrangian* if $L^\perp = L$, in which case we say that $[P, q]$ is *metabolic*. In the context of quadratic forms on projective modules, every metabolic object is in fact hyperbolic, that is, of the form $\mathrm{Hyp}(Q) := (Q^\vee \oplus Q, q_{\text{hyp}})$ for Q a finitely generated projective module, where $q_{\text{hyp}}(f, v) = f(v)$, though this will no long be true for other types of forms, such as symmetric bilinear forms. The quadratic *Witt group* is then defined as the quotient

$$\mathrm{GW}^{\mathrm{q}}_0 \twoheadrightarrow \mathrm{W}^{\mathrm{q}}_0(R)$$

by the metabolic/hyperbolic classes. One can also replace quadratic forms by symmetric forms to obtain the corresponding symmetric Grothendieck-Witt and Witt groups. There are also many other flavours of forms one can use, such as anti-symmetric forms and even forms (those admitting an unspecified quadratic refinement).

Using techniques and inspiration from algebraic K-theory, Grothendieck-Witt and Witt groups were defined and studied in a variety of contexts:

i) Karoubi-Villamayor: Higher Grothendieck-Witt groups of rings with involution.
ii) Knebusch: Zero'th Witt group for exact categories with duality/schemes.
iii) Schlichting: Higher Grothendieck-Witt groups of exact categories with duality/schemes.
iv) Balmer: Higher Witt groups of triangulated $\mathbb{Z}[1/2]$-linear categories, schemes over $\mathbb{Z}[1/2]$.

In a parallel development, L-groups were introduced by Wall and extensively studied by Ranicki in geometric topology; these play a crucial role in surgery theory and the classification of manifolds, and involve quadratic forms on complexes of modules over group rings, where the group in question is often the fundamental group of a given space. More generally, one could define quadratic and symmetric L-groups for any ring with involution, and also in some more general situations.

For rings with involution over which 2 is invertible, the quadratic and symmetric L-groups $\mathrm{L}^{\mathrm{q}}_0(R)$ and $\mathrm{L}^{\mathrm{s}}_0(R)$ are isomorphic to the quadratic and symmetric Witt groups, and similarly the higher L-groups are isomorphic to Balmer's higher Witt groups. On the other hand, when 2 is not invertible the two groups differ already in degree 0. One manner in which the two approaches can be bridged is by setting up the theory using Poincaré ∞-categories. This notion was introduced by Lurie in his lecture notes on L-theory, and was developed in a series of joint papers with Baptiste Calmès, Emanuele Dotto, Fabian Hebestreit, Markus Land, Kristian Moi, Denis Nardin, Thomas Nikolaus and Wolfgang Steimle [CDH+23, CDH+25, CDH+26].

4.0.1. **Definition.** A *Poincaré ∞-category* is a pair $(\mathcal{C}, \Omega)$ where $\mathcal{C}$ is a stable ∞-category and $\Omega : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}p$ is a functor satisfying the following properties:

i) The cross-effect $\mathrm{B}_{\Omega}(x, y) = \mathrm{fib}[\Omega(x \oplus y) \to \Omega(x) \otimes \Omega(y))]$ is of the form $\mathrm{B}_{\Omega}(x, y) = \mathrm{map}(x, \mathrm{D}y)$ for some equivalence $\mathrm{D} : \mathcal{C} \to \mathcal{C}^{\mathrm{op}}$ (which is then uniquely determined by B_{Ω}).
ii) The cofibre of the induced map $\mathrm{B}_{\Omega}(x, x)_{\mathrm{hC}_2} \to \Omega(x)$ is an exact functor in x.

We refer to Ω as a Poincaré structure on $\mathcal{C}$. One may view $(\mathcal{C}, \Omega)$ as a categorification of a the notion of a non-degenerate quadratic form, so that instead of a module we have a stable ∞-category and instead of a non-degenerate quadratic form we have a quadratic functor satisfying a suitable non-degeneracy condition. The functor Ω should be considered as an axiomatization of the association $P \mapsto \mathrm{Qdr}(P)$, namely, of the functor associating to a module the abelian group of quadratic forms on it (only that instead of an abelian group, we now have a spectrum of such). This flexibility then also enables one to specify in the input what flavour forms one is considering. For example, even when $\mathcal{C}$, is, say, the perfect derived category of a commutative ring R, one could consider different Poincaré structures encoding different flavours of forms: quadratic, symmetric bilinear, anti-symmetric, etc., without needing to change the theory in any way. We shall hence generically refer to points $q \in \Omega^{\infty}(x)$ as *hermitian forms* on x, and to pairs (x, q) as *hermitian objects*. Any hermitian form q determines a point in $\Omega^{\infty}\mathrm{B}_{\Omega}(x, x)$ and hence a map $q_{\sharp} : x \to \mathrm{D}x$. We then say that the hermitian form q is *Poincaré* if $q_{\sharp}$ is an equivalence, in which case we call (x, q) a *Poincaré object*. Poincaré forms then correspond to non-degenerate quadratic forms in this abstract setting. The collection of Poincaré objects (x, q) can then be organized into a space (or an ∞-groupoid), which we denote by $\mathrm{Pn}(\mathcal{C}, \Omega)$.

Given two Poincaré objects $(x, q), (x', q')$, a cobordism from (x, q) to (y, q') is a diagram of hermitian objects

$$\begin{tikzcd} & (w, q'') \arrow[dl, "g"'] \arrow[dr, "f"] & \\ (x, q) & & (y, q') \end{tikzcd}$$

such that the induced map $\mathrm{cof}(g) \to \mathrm{D}\,\mathrm{fib}(f)$ is an equivalence. Cobordisms can be composed, forming a hermitian analogue of Quillen's Q-construction, and which we shall denote by $\mathrm{Q}^{\mathrm{h}}(\mathcal{C}, \Omega)$. This construction was already used by Schlichting to define the higher Grothendieck-Witt groups of an exact category with duality. We say that two Poincaré objects are cobordant if there exists a cobordism between them. Unlike the case of exact categories where surjectivity/injectivity conditions were imposed making the cobordism a-symmetric, here the notion is completely symmetric for (x, q) and (y, q'). As a result, being cobordant is an equivalence relation on Poincaré objects. We refer to the equivalence classes of this relation as cobordism classes. The L-group $\mathrm{L}_0(\mathcal{C}, \Omega)$ is then defined as the group of cobordism classes of Poincaré objects with the group structure given by $[x, q] + [x', q'] = [x \oplus x', q \oplus q']$.

In his lecture notes Lurie defines the L-space of a Poincaré ∞-category as follows. For $n \geq 0$, let $\mathcal{T}_n$ denote the opposite of the poset of non-empty subsets of $[n]$ (so that maps in $\mathcal{T}_n$ go from big subsets to small). Given a Poincaré ∞-category $(\mathcal{C}, \Omega)$ we may then define a new Poincaré ∞-category $(\mathcal{C}, \Omega)^{\mathcal{T}_n}$ whose objects are the diagrams $\varphi : \mathcal{T}_n \to \mathcal{C}$ and whose Poincaré structure is the defined by

$$\Omega^{\mathcal{T}_n}(\varphi) = \lim_{S \in \mathcal{T}_n} \Omega(\varphi(S)).$$

Since the poset $\mathcal{T}_n$ depends functorially on $[n]$ (via taking images of subsets) we obtain a simplicial object in spaces $\mathrm{Pn}((\mathcal{C}, \Omega)^{\mathcal{T}_n})$, and we define the L-theory space

$$\mathcal{L} = |\mathrm{Pn}((\mathcal{C}, \Omega)^{\mathcal{T}_n}|$$

as the corresponding geometric realization. This is again an E_∞-space, and more precisely an E_∞-group. The higher L-groups $\mathrm{L}_n(\mathcal{C}, \Omega)$ are then given by the homotopy groups

$$\mathrm{L}_n(\mathcal{C}, \Omega) := \pi_n \mathcal{L}(\mathcal{C}, \Omega).$$

The notion of Verdier sequences of stable ∞-categories extends to the Poincaré context, giving what we call Poincaré-Verdier sequences. These are all Verdier sequences on underlying stable ∞-categories with additional conditions satisfied by the Poincaré structures.

4.0.2. **Proposition.** *The functor $\mathcal{L}$ sends Poincaré-Verdier sequences to fibre sequences.*

An example of a Verdier sequence among stable ∞-categories to which the above proposition can be applied is the metabolic sequence

$$(\mathcal{C}, \Omega^{[-1]}) \to \mathrm{Met}(\mathcal{C}, \Omega) \xrightarrow{\mathrm{met}} (\mathcal{C}, \Omega)$$

where $\mathrm{Met}(\mathcal{C}, \Omega)$ is the Poincaré ∞-category whose objects are maps $x \to w$ and whose Poincaré structure is given by $\Omega_{\mathrm{met}}(x \to w) = \mathrm{fib}[\Omega(x) \to \Omega(w)]$ and $\Omega^{[-1]} = \Omega\Omega$. The Poincaré functor met is the given by $[x \to w] \mapsto x$ with the natural map $\Omega_{\mathrm{met}}(x \to w) \to \Omega(x)$ and its fibre is the full subcategory of the form $w \to 0$, on which the Poincaré structure is given by $\mathrm{fib}[0 \to \Omega(w)] = \Omega\Omega(w)$. Since $\mathcal{L}$ is Verdier localising it sends the metabolic sequence to a fibre sequence of spaces. At the same time, one can show that $\mathcal{L}(\mathrm{Met}(\mathcal{C}, \Omega)) \simeq *$, so that one obtains that $\mathcal{L}(\mathcal{C}, \Omega^{[-1]}) \simeq \Omega\, \mathcal{L}(\mathcal{C}, \Omega)$. One result of this is that $\mathrm{L}_n(\mathcal{C}, \Omega) = \mathrm{L}_0(\mathcal{C}, \Omega^{[-n]})$, so that one can describe $\mathrm{L}_n(\mathcal{C}, \Omega)$ concretely as the group of cobordism classes of Poincaré objects with respect to the $(-n)$-shifted Poincaré structure $\Omega^{[-n]}$. Another result is that one can defined the L-spectrum via the sequence of spaces

$$\mathcal{L}(\mathcal{C}, \Omega), \mathcal{L}(\mathcal{C}, \Omega^{[1]}), \mathcal{L}(\mathcal{C}, \Omega^{[2]}), \ldots$$

together with the equivalences $\mathcal{L}(\mathcal{C}, \Omega^{[n]}) \simeq \Omega\, \mathcal{L}(\mathcal{C}, \Omega^{[n+1]})$.

In [CDH+25] we have also extended the definition of the GW-space via the hermitian Q-construction to Poincaré ∞-categories by setting

$$\mathcal{GW}(\mathcal{C}, \Omega) = |\, \mathrm{Q}^{\mathrm{h}}(\mathcal{C}, \Omega^{[1]})|$$

where $\mathrm{Q}^{\mathrm{h}}(-)$ is the cobordism category, or hermitian Q-construction for Poincaré ∞-categories. One can show that this is equivalent to $\mathrm{fib}[|\, \mathrm{Q}^{\mathrm{h}}(\mathcal{C}, \Omega)| \to |\, \mathrm{Q}(\mathcal{C})|]$, in analogy with the definition of the GW-space in the setting of exact categories. Using an iterated version of the Q-construction we also define the GW-spectrum. The GW- and L-spectrum are then related by a fundamental exact sequence

$$\mathrm{K}(\mathcal{C}, \Omega)_{\mathrm{hC}_2} \to \mathrm{GW}(\mathcal{C}, \Omega) \to \mathrm{L}(\mathcal{C}, \Omega)$$

where the first term is the homotopy orbits of the C_2-action on $\mathrm{K}(\mathcal{C})$ induced by the duality. Such a sequence was previous constructed by Schlichting for $\mathbb{Z}[1/2]$-linear dg-categories with duality. In particular, since K and L are Poincaré-Verdier localising it follows that GW is Poincaré-Verdier localising as well.

4.0.3. **Examples.** For a commutative ring $\boldsymbol{R}$, four of the Poincaré structures one can put on $\mathcal{D}^{\mathrm{p}}(\boldsymbol{R})$ are of special interest. They are related by a sequence of maps of Poincaré structures

$$\Omega^{\mathrm{q}} \Rightarrow \Omega^{\mathrm{gq}} \Rightarrow \Omega^{\mathrm{gs}} \Rightarrow \Omega^{\mathrm{s}}$$

and we call them the quadratic, genuine quadratic, genuine symmetric, and symmetric Poincaré structures, respectively. The quadratic and symmetric ones are given by the explicit formulas

$$\Omega^{q}(M) := \mathrm{map}_R(M \otimes_R M, R)_{\mathrm{hC}_2} \quad \text{and} \quad \Omega^{s}(M) := \mathrm{map}_R(M \otimes_R M, R)^{\mathrm{hC}_2},$$

for $M \in \mathcal{D}^{\mathrm{p}}(R)$. The genuine quadratic an genuine symmetric are, on the other hand, obtained by suitably extending from $\mathrm{Proj}(R)$ to $\mathcal{D}^{\mathrm{p}}(R)$ the functors $P \mapsto \mathrm{Qdr}(P)$ and $P \mapsto \mathrm{Sym}(P)$ viewed as spectrum valued functors on $\mathrm{Proj}(R)^{\mathrm{op}}$ via the Eilenberg-MacLane embedding $\mathcal{A}b \to \mathcal{S}p$.

A theorem of Hebestreit and Steimle then asserts that the Grothendieck-Witt spaces of $(\mathcal{D}^{\mathrm{p}}(R), \Omega^{\mathrm{gs}})$ and $(\mathcal{D}^{\mathrm{p}}(R), \Omega^{\mathrm{gq}})$ coincide with the quadratic and symmteric Grothendieck-Witt spaces defined by Karoubi and Villamayor. These examples can also be extended to schemes, in which case we can compare the corresponding genuine symmetric/quadratic Grothendieck-Witt spaces with those defined by Schlichting, at least for divisorial schemes. On the other hand, the quadratic and symmetric L-groups of Wall and Ranicki correspond to the L-groups of the Poincaré structures Ω^{q} and Ω^{s}, respectively, explaining why the two branches of the theory could not be reconciled with 2 is not invertible.

5. Hermitian Trace methods

Considering GW-theory and L-theory as suitable cohomology theories for schemes, it is natural to ask if one can extend the trace methods of K-theory to these invariants. In what follows we will restrict attention to L-theory, though a comprehensive treatment of these questions encompassing both GW and L is currently in process with Thomas Nikolaus and Jay Shah [HTS26].

The analogue of topological Hochschild homology for L-theory is the functor we call geometric Hochschlid homology:

$$\mathrm{THG}(\mathcal{C}, \Omega) := \operatorname*{colim}_{[x \to y] \in \mathrm{TwAr}^{\mathrm{r}}(\mathcal{C})} \Lambda(\mathrm{D}x) \otimes \Lambda(y)$$

where $\Lambda(x) := \mathrm{cof}[\mathrm{B}(x,x)_{\mathrm{hC}_2} \to \Omega(x)]$ is the linear part (or approximation) of Ω. Let us describe two equivalent manners of writing this expression. Let

$$\tilde{\Lambda}: \ \mathrm{Ind}(\mathcal{C}^{\mathrm{op}}) \to \mathcal{S}p \quad \text{and} \quad \tilde{\mathrm{B}}: \ \mathrm{Ind}(\mathcal{C}^{\mathrm{op}}) \times \mathrm{Ind}(\mathcal{C}^{\mathrm{op}}) \to \mathcal{S}p$$

be the left Kan extensions of $\Lambda: \ \mathcal{C}^{\mathrm{op}} \to \mathcal{S}p$ and $\mathrm{B}: \ \mathcal{C}^{\mathrm{op}} \times \mathcal{C}^{\mathrm{op}} \to \mathcal{S}p$, where $\mathrm{Ind}(\mathcal{C}^{\mathrm{op}}) := \mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}, \mathcal{S}p)$ is the Ind-completion of $\mathcal{C}^{\mathrm{op}}$. Given $\Lambda' \in \mathrm{Ind}(\mathcal{C}^{\mathrm{op}})$ we then have

$$\widetilde{\Lambda}(\Lambda') = \operatorname*{colim}_{[x \to y] \in \mathrm{TwAr}^{\mathrm{r}}(\mathcal{C})} \Lambda'(x) \otimes \Lambda(y),$$

and so we can write

$$\mathrm{THG}(\mathcal{C}, \Omega) = \tilde{\Lambda}(\mathrm{D}\Lambda),$$

where $\mathrm{D}\Lambda \in \mathrm{Ind}(\mathcal{C}^{\mathrm{op}})$ is the functor $x \mapsto \Lambda(\mathrm{D}x)$, and is also the image of Λ under the equivalence $\mathrm{Ind}(\mathcal{C}) \to \mathrm{Ind}(\mathcal{C}^{\mathrm{op}})$ induced by D. Now since $\mathrm{B}(-, x) = \mathrm{map}(-, \mathrm{D}x)$ we have $\tilde{\mathrm{B}}(-, \mathrm{D}\Lambda) = \mathrm{map}(-, \Lambda) = \Lambda(-)$ and so we can also write

$$\mathrm{THG}(\mathcal{C}, \Omega) = \tilde{\mathrm{B}}(\mathrm{D}\Lambda, \mathrm{D}\Lambda).$$

This formula shows in particular that $\mathrm{THG}(\mathcal{C}, \Omega)$ carries a natural μ_2-action (where μ_2 is C_2, but viewed as the 2-torsion subgroup of S^1). Now we have a natural map

$$\Lambda(x) = \mathrm{cof}[\mathrm{B}(x,x)_{\mathrm{hC}_2} \to \Omega(x)] \to \mathrm{cof}[\mathrm{B}(x,x)_{\mathrm{hC}_2} \to \mathrm{B}(x,x)^{\mathrm{hC}_2}] = \mathrm{B}(x,x)^{\mathrm{tC}_2},$$

which determines a natural map $\tilde{\Lambda}(\mathrm{D}\Lambda) \to \tilde{(}\mathrm{D}\Lambda, \mathrm{D}\Lambda)^{\mathrm{tC}_2}$, and hence a "Frobenius map"

$$F: \ \mathrm{THG}(\mathcal{C}, \Omega) \to \mathrm{THG}(\mathcal{C}, \Omega)^{\mathrm{t}\mu_2}$$

We may view this as a mini-version of the cyclotomic structure carried by THH. We call this a μ_2-cyclotomic structure. Let me now explain how to construct a trace map from L-theory to THG which, as was the case for the trace map from K-theory, is compatible with the μ_2-cyclotomic structure of THG. Note first that the collection of μ_2-cyclotomic spectra can be organized into an ∞-category CycSp^{μ_2}. Now every spectrum E can be viewed as a μ_2-cyclotomic spectrum in a trivial manner by endowing E with the trivial μ_2-action and setting the structure map to be the composite

$$E \to E^{\mathrm{h}\mu_2} \to E^{\mathrm{t}\mu_2},$$

where the first map arises from the fact that the μ_2-action is trivial and the second map is the natural map from homotopy fixed points to Tate. Forming trivial μ_2-cyclotomic structures then yields a functor

$$\mathrm{Triv}\colon \mathcal{S}p \to \mathrm{CycSp}^{\mu_2},$$

which admits a right adjoint we call

$$\mathrm{TC}^{\mu_2}\colon \mathrm{CycSp}^{\mu_2} \to \mathcal{S}p.$$

Constructing a trace map $\mathrm{L}(\mathcal{C},\Omega) \to \mathrm{THG}(\mathcal{C},\Omega)$ that is compatible with the μ_2-cyclotomic structure is then equivalent to constructing a map $\mathrm{Triv}(\mathrm{L}(\mathcal{C},\Omega)) \to \mathrm{THG}(\mathcal{C},\Omega)$, or, by adjunction, a map $\mathrm{L}(\mathcal{C},\Omega) \to \mathrm{TC}^{\mu_2}(\mathrm{THG}(\mathcal{C},\Omega))$.

5.0.1. **Lemma.** TC^{μ_2} *of the* μ_2*-cyclotomic spectrum* $\mathrm{THG}(\mathcal{C},\Omega)$ *can be expressed as any of the following equivalent equalized formulas:*

$$\begin{aligned}\mathrm{TC}^{\mu_2}(\mathrm{THG}(\mathcal{C},\Omega)) =&\mathrm{Eq}\left[\tilde{\mathrm{B}}(\mathrm{D}\Lambda,\mathrm{D}\Lambda)^{\mathrm{hC}_2} \rightrightarrows \tilde{\mathrm{B}}(\mathrm{D}\Lambda,\mathrm{D}\Lambda)^{\mathrm{tC}_2}\right]\\ =&\mathrm{Eq}\left[\tilde{\Omega}(\mathrm{D}\Lambda) \rightrightarrows \tilde{\Lambda}(\mathrm{D}\Lambda)\right]\\ =&\mathrm{Eq}\left[\tilde{\mathrm{B}}(\mathrm{D}\Lambda,\mathrm{D}\Lambda) \rightrightarrows \tilde{\mathrm{B}}^{[1-\sigma]}(\mathrm{D}\Lambda,\mathrm{D}\Lambda)_{\mathrm{hC}_2}\right]\end{aligned}$$

where:

- *In first equalizer the two maps are the canonical one and the composite* $\mathrm{B}(\mathrm{D}\Lambda,\mathrm{D}\Lambda)^{\mathrm{hC}_2} \to \mathrm{B}(\mathrm{D}\Lambda,\mathrm{D}\Lambda) = \tilde{\Lambda}(\mathrm{D}\Lambda) \to \tilde{\mathrm{B}}(\mathrm{D}\Lambda,\mathrm{D}\Lambda)^{\mathrm{tC}_2}$.
- *In the second equalizer the two maps come from the natural transformations* $\tilde{\Omega}(x) \Rightarrow \tilde{\Lambda}(x)$ *and* $\tilde{\Omega}(x) \Rightarrow \tilde{\mathrm{B}}(x,x)$ *evaluated at* $x = \mathrm{D}\Lambda$*, under the equivalence* $\tilde{\mathrm{B}}(\mathrm{D}\Lambda,\mathrm{D}\Lambda) = \tilde{\Lambda}(\mathrm{D}\Lambda)$*.*
- *In the third equalizer the two maps are the canonical projection to the quotient and the composite*

$$\tilde{\mathrm{B}}(\mathrm{D}\Lambda,\mathrm{D}\Lambda) = \tilde{\Lambda}(\mathrm{D}\Lambda) \to \tilde{\mathrm{B}}(\mathrm{D}\Lambda,\mathrm{D}\Lambda)^{\mathrm{tC}_2} \simeq \tilde{\mathrm{B}}^{[1-\sigma]}(\mathrm{D}\Lambda,\mathrm{D}\Lambda)^{\mathrm{tC}_2} \to \tilde{\mathrm{B}}^{[1-\sigma]}(\mathrm{D}\Lambda,\mathrm{D}\Lambda)_{\mathrm{hC}_2}.$$

In order to construct the trace map for L-theory it will hence suffice to construct a natural map from L to any of the above equivalent equalizers. Our construction will make use of the following notion, due to Ranicki:

5.0.2. **Definition.** A *normal hermitian object* is a tuple (y,q,ℓ,α) where $y \in \mathcal{C}$ is an object, $q \in \Omega^\infty \Omega(y)$ is a hermitian form on y (determining in particular a map $q_\sharp\colon y \to \mathrm{D}y$), $\ell \in \Omega^\infty\Lambda(\mathrm{D}y)$ is a linear form on $\mathrm{D}y$, and α is a homotopy between $q_\sharp^*\ell \in \Omega^\infty\Lambda(y)$ and the image of q under the map $\Omega^\infty\Omega(y) \to \Omega^\infty\Lambda(y)$. The collection of normal hermitian objects can be organized into an ∞-category, which we denote by $\mathrm{He}^{\mathrm{nor}}(\mathcal{C},\Omega)$.

Now we have a forgetful functor

$$\mathrm{He}^{\mathrm{nor}}(\mathcal{C},\Omega) \to (\mathcal{C}_{/\Lambda})^{\mathrm{op}} \qquad (y,q,\ell,\alpha) \mapsto (\mathrm{D}y,\ell),$$

which one can show is a right fibration, classified by the functor

$$(\mathcal{C}_{/\Lambda})^{\mathrm{op}} \to \mathcal{S} \qquad (x,\ell) \mapsto \mathrm{Eq}\left[\Omega^\infty\Omega(\mathrm{D}x) \rightrightarrows \Omega^\infty\Lambda(\mathrm{D}x)\right],$$

where for $(x,\ell) \in \mathcal{C}_{/\Lambda}$ the two maps in the equalizer are

- The canonical one given by evaluating the natural transformation $\Omega^\infty\Omega \Rightarrow \Omega^\infty\Lambda$ at $\mathrm{D}x$.
- The map sending $q \in \Omega^\infty\Omega(\mathrm{D}x)$ to $q_\sharp^*\ell$, where $q_\sharp\colon \mathrm{D}x \to x$ is the map determined by q.

This means that

$$\begin{aligned}|\mathrm{He}^{\mathrm{nor}}(\mathcal{C},\Omega)| =& \operatorname*{colim}_{(x,\ell)\in\mathcal{C}_{/\Lambda}} \mathrm{Eq}\left[\Omega^\infty\Omega(\mathrm{D}x) \rightrightarrows \Omega^\infty\Lambda(\mathrm{D}x)\right]\\ =&\mathrm{Eq}\left[\Omega^\infty\tilde{\Omega}(\mathrm{D}\Lambda) \rightrightarrows \Omega^\infty\tilde{\Lambda}(\mathrm{D}\Lambda)\right]\\ =&\Omega^\infty\mathrm{Eq}\left[\tilde{\Omega}(\mathrm{D}\Lambda) \rightrightarrows \tilde{\Lambda}(\mathrm{D}\Lambda)\right].\end{aligned}$$

By the previous lemma, we thus obtain a natural equivalence

$$|\mathrm{He}^{\mathrm{nor}}(\mathcal{C},\Omega)| \simeq \Omega^\infty\mathrm{TC}^{\mu_2}(\mathrm{THG}(\mathcal{C},\Omega)).$$

We now construct a map from the L-space $\mathcal{L}$ to the above equivalent spaces. For this, note that if (x,q) is a Poincaré object then $q_\sharp\colon x \to \mathrm{D}x$ is an equivalence and hence (x,q) extends to a normal hermitian object (x, q, ℓ, α) in an essentially unique manner. In particular, the core ∞-groupoid $\mathrm{He}^{\mathrm{nor}}(\mathcal{C}, \Omega)^{\simeq}$ contains $\mathrm{Pn}(\mathcal{C}, \Omega)$ as a full subgroupoid. This observation leads to a natural map of spaces

$$\mathrm{Pn}(\mathcal{C}, \Omega) \to |\mathrm{He}^{\mathrm{nor}}(\mathcal{C}, \Omega)|.$$

Now recall that the L-space was defined as the geometric realization of the simplicial object $\mathrm{Pn}((\mathcal{C}, \Omega)^{\mathcal{T}_\bullet})$.

5.0.3. **Lemma.** *We have a natural equivalence*

$$\mathrm{He}^{\mathrm{nor}}((\mathcal{C}, \Omega)^{\mathcal{T}_n}) \simeq \mathrm{Fun}(\mathcal{T}_n, \mathrm{He}^{\mathrm{nor}}(\mathcal{C}, \Omega)).$$

Since each $\mathcal{T}_n$ has an initial object this means in particular that $|\mathrm{He}^{\mathrm{nor}}((\mathcal{C}, \Omega)^{\mathcal{T}_n})| \simeq |\,\mathrm{Fun}(\mathcal{T}_n, \mathrm{He}^{\mathrm{nor}}(\mathcal{C}, \Omega))| \simeq |\mathrm{He}^{\mathrm{nor}}((\mathcal{C}, \Omega)|$ for every n, in other words, the simplicial object $|\mathrm{He}^{\mathrm{nor}}((\mathcal{C}, \Omega)^{\mathcal{T}_n})|$ is homotopy constant. The natural transformation $\mathrm{Pn}(-) \Rightarrow |\mathrm{He}^{\mathrm{nor}}(-)|$ hence induces a map

$$\mathcal{L}(\mathcal{C}, \Omega) = |\mathrm{Pn}((\mathcal{C}, \Omega)^{\mathcal{T}_\bullet})| \to ||\mathrm{He}^{\mathrm{nor}}((\mathcal{C}, \Omega)^{\mathcal{T}_\bullet})|| = |\mathrm{He}^{\mathrm{nor}}(\mathcal{C}, \Omega)| = \Omega^\infty \mathrm{TC}^{\mu_2}(\mathrm{THG}(\mathcal{C}, \Omega))$$

which we can refine to a spectrum level map

$$\mathrm{L}(\mathcal{C}, \Omega) \to \mathrm{TC}^{\mu_2}(\mathrm{THG}(\mathcal{C}, \Omega))$$

using the fact that both sides are localising and vanish on metabolic categories, so that both side translate shifting Ω to shifting outside. We thus obtain the μ_2-cyclotomic trace map.

5.0.4. **Definition.** Let $(\mathcal{C}, \Omega)$ be a Poincaré ∞-category with associated duality D, and let $\Omega^{\mathrm{q}}_{\mathrm{D}}(x) = \mathrm{map}(x, \mathrm{D}x)_{\mathrm{hC}_2}$ be the associated quadratic Poincaré structure. The cofibre

$$\mathrm{L}^{\mathrm{nor}}(\mathcal{C}, \Omega) := \mathrm{cof}[\mathrm{L}(\mathcal{C}, \Omega^{\mathrm{q}}_{\mathrm{D}}) \to \mathrm{L}(\mathcal{C}, \Omega)]$$

of the natural map is then called *normal L-theory*.

Since quadratic Poincaré structures have trivial linear parts we have $\mathrm{THG}(\mathcal{C}, \Omega^{\mathrm{q}}_{\mathrm{D}}) = 0$ and so the μ_2-cyclotomic trace map factors through a map

$$\mathrm{L}^{\mathrm{nor}}(\mathcal{C}, \Omega) \to \mathrm{TC}^{\mu_2}(\mathcal{C}, \Omega).$$

5.0.5. **Theorem** ([HTS26])**.** *The above map is an equivalence.*

Sketch. Let $\mathrm{Ar}^{\mathrm{nor}}(\mathcal{C}, \Omega)$ be the Poincaré ∞-category whose underlying stable ∞-category is the arrow ∞-category $\mathrm{Ar}(\mathcal{C}) = \mathrm{Fun}(\Delta^1, \mathcal{C})$ and whose Poincaré structure is given by the formula

$$\Omega^{\mathrm{nor}}(x \to y) = \Lambda(y) \times_{\Lambda(x)} \Omega(x) \times_{\mathrm{B}(x,x)} \mathrm{B}(x, y).$$

Note that Ω^{nor} restricts to Ω on identity arrows $x = x$ and that $\Omega^{\mathrm{nor}}(x \to y)$ receives a map from $\Omega^{\mathrm{q}}_{\mathrm{ar}}(x) := \Omega^{\mathrm{q}}_{\mathrm{D}} \times_{\mathrm{B}(x,x)} \mathrm{B}(x, y)$, so that we can form a commutative square of Poincaré ∞-categories

$$\begin{tikzcd}
(\mathcal{C}, \Omega^{\mathrm{q}}_{\mathrm{D}}) \arrow[r] \arrow[d, "\mathrm{id}_{(-)}"'] & (\mathcal{C}, \Omega) \arrow[d] \\
\mathrm{Ar}(\mathcal{C}, \Omega^{\mathrm{q}}_{\mathrm{D}}) \arrow[r] & \mathrm{Ar}^{\mathrm{nor}}(\mathcal{C}, \Omega),
\end{tikzcd}$$

where $\mathrm{Ar}(\mathcal{C}, \Omega^{\mathrm{q}}_{\mathrm{D}}) = (\mathrm{Ar}(\mathcal{C}), \Omega^{\mathrm{q}}_{\mathrm{ar}})$. Here the horizontal arrows are equivalences on underlying stable ∞-categories (that is, they only change the Poincaré structure) and the vertical maps are Poincaré-Verdier inclusions. One can then show that this square is a pushout square in $\mathrm{Cat}^{\mathrm{p}}_{\infty}$, so that the induced map on vertical cofibres is an equivalence. This implies that L sends this square to an exact square. At the same time, L vanishes on $\mathrm{Ar}(\mathcal{C}, \Omega^{\mathrm{q}}_{\mathrm{D}})$, and so $\mathrm{L}(\mathrm{Ar}^{\mathrm{nor}}(\mathcal{C}, \Omega)) = \mathrm{L}^{\mathrm{nor}}(\mathcal{C}, \Omega)$. This means that

$$\Omega^\infty \mathrm{L}^{\mathrm{nor}}(\mathcal{C}, \Omega) = \mathcal{L}(\mathrm{Ar}^{\mathrm{nor}}(\mathcal{C}, \Omega)) = |\mathrm{Pn}((\mathrm{Ar}^{\mathrm{nor}}(\mathcal{C}, \Omega))^{\mathcal{T}_\bullet})| = |\mathrm{Pn}(\mathrm{Ar}^{\mathrm{nor}}((\mathcal{C}, \Omega)^{\mathcal{T}_\bullet}))|.$$

One can then to show that there is a natural equivalence $\mathrm{Pn}(\mathrm{Ar}^{\mathrm{nor}}(\mathcal{C}, \Omega)) \simeq \mathrm{He}^{\mathrm{nor}}(\mathcal{C}, \Omega)^{\simeq}$ and so

$$|\mathrm{Pn}(\mathrm{Ar}^{\mathrm{nor}}((\mathcal{C}, \Omega)^{\mathcal{T}_\bullet}))| \simeq |\mathrm{He}^{\mathrm{nor}}((\mathcal{C}, \Omega)^{\mathcal{T}_\bullet})^{\simeq}| \simeq |\,\mathrm{Fun}(\mathcal{T}_\bullet, \mathrm{He}^{\mathrm{nor}}(\mathcal{C}, \Omega))^{\simeq}| \simeq |\mathrm{He}^{\mathrm{nor}}(\mathcal{C}, \Omega)|.$$

Assembling all this we conclude that the map

$$\mathrm{L}^{\mathrm{nor}}(\mathcal{C}, \Omega) \to \mathrm{TC}^{\mu_2}(\mathcal{C}, \Omega)$$

is an equivalence on infinite loop spaces, and hence an equivalence since both sides translate shifting Ω to shifts on the outside. $\square$

5.0.6. **Corollary.** *The fibre of the μ_2-cyclotomic trace map* $\mathrm{L} \Rightarrow \mathrm{TC}^{\mu_2}$ *is invariant under nilpotent extensions.*

Proof. This fibre is quadratic L-theory, which is known to be invariant under nilpotent extensions by work of Wall (a proof is recorded in [CDH+26, Proposition 2.3.7]). □

5.0.7. **Example** (Normal L-theory of the sphere)**.** For $(\mathcal{C}, \Qoppa) = (\mathcal{S}p^{\mathrm{f}}, \Qoppa^{\mathrm{u}})$, using the third equalizer in Lemma 5.0.1 yields an identification

$$\mathrm{L}^{\mathrm{nor}}(\mathcal{S}p^{\mathrm{f}}, \Qoppa^{\mathrm{u}}) \simeq \mathrm{Eq}[\mathbb{S} \rightrightarrows \mathbb{S}^{1-\sigma}_{\mathrm{hC}_2}].$$

Under this equivalence, the projection from the equalizer to the first term coincides with the map $\mathrm{L}(\mathcal{S}p^{\mathrm{f}}, \Qoppa^{\mathrm{u}}) \to \mathrm{THG}(\mathcal{S}p^{\mathrm{f}}, \Qoppa^{\mathrm{u}}) = \mathbb{S}$, which admits a section given by the unit map. It then follows that the maps in the equalizer are homotopic to each other, yielding an equivalence

$$\mathrm{L}^{\mathrm{nor}}(\mathcal{S}p^{\mathrm{f}}, \Qoppa^{\mathrm{u}}) \simeq \mathbb{S} \oplus \mathbb{S}^{-\sigma}_{\mathrm{hC}_2} \simeq \mathbb{S} \oplus \mathrm{MTO}(1).$$

This formula for normal L-theory of the sphere was first proven by Weiss and Williams in [WW14].

5.0.8. **Example.** Let X be a space equipped with a spherical fibration ξ, and take $(\mathcal{C}, \Qoppa) = (\mathcal{S}p^{\mathrm{f}}_X, \Qoppa^{\mathrm{v}}_{\xi})$ to be the ∞-category of perfect parameterized spectra over X equipped with the associated visible Poincaré structure. Then the linear part of $\Qoppa^{\mathrm{v}}_{\xi}$ is represented by ξ and one can show that for $\mathcal{F}, \mathcal{G} \in \mathcal{S}p^{\mathrm{f}}_X$ there is a natural equivalence

$$\mathrm{B}_{\xi}(\mathrm{D}_{\xi}\mathcal{F}, \mathrm{D}_{\xi}\mathcal{G}) \simeq \operatorname*{colim}_{x \in X} \xi(x)^{-1} \otimes \mathcal{F}(x) \otimes \mathcal{G}(x).$$

The normal L-theory formula (using the third equalizer in Lemma 5.0.1) then gives an identification

$$\begin{aligned}\mathrm{L}^{\mathrm{nor}}(\mathcal{C}, \Qoppa^{\mathrm{v}}_{\xi}) &= \mathrm{Eq}[\tilde{\mathrm{B}}_X(\mathrm{D}\xi, \mathrm{D}\xi) \rightrightarrows \tilde{\mathrm{B}}^{[1-\sigma]}(\mathrm{D}\xi, \mathrm{D}\xi)_{\mathrm{hC}_2}] \\ &= \mathrm{Eq}\left[\operatorname*{colim}_{x\in X} \xi(x) \rightrightarrows \operatorname*{colim}_{x \in X}(\mathbb{S}^{1-\sigma} \otimes \xi(x))_{\mathrm{hC}_2}\right] \\ &= \operatorname*{colim}_{x\in X} \mathrm{Eq}[\xi(x) \to (\mathbb{S}^{1-\sigma} \otimes \xi(x))_{\mathrm{hC}_2}].\end{aligned}$$

In particular, if we denote by $\mathbb{S}_X$ the constant spherical fibration on X, then the association $X \mapsto \mathrm{L}^{\mathrm{nor}}(\mathcal{C}, \Qoppa^{\mathrm{v}}_{\mathbb{S}_X}) \simeq X_+ \wedge \mathrm{L}^{\mathrm{nor}}(\mathcal{S}p^{\mathrm{f}}, \Qoppa^{\mathrm{u}})$ is a colimit preserving functor from spaces to spectra.

5.0.9. **Example.** Let A be an $\mathcal{E}_1$-ring spectrum with involution and $(A, N, \alpha : N \to A^{\mathrm{tC}_2})$ a genuine refinement of A as a module with involution over itself. The third equalizer formula then gives an identification

$$\mathrm{L}(\mathrm{Mod}^{\omega}(A), \Qoppa^{\mathrm{q}}, \Qoppa^{\alpha}_A) = \mathrm{Eq}\big[\mathrm{map}_A(\mathrm{D}N, N) \rightrightarrows \tilde{\mathrm{B}}^{[1-\sigma]}(\mathrm{D}N, \mathrm{D}N)_{\mathrm{hC}_2}\big] = \mathrm{Eq}\big[N \otimes_A N \rightrightarrows (\mathbb{S}^{1-\sigma} \otimes N \otimes_A N)_{\mathrm{hC}_2}\big].$$

Here, to form the tensor product $N \otimes_A N$ we turn the left action on the first N factor to a right action using the involution on A.

The above formula for normal L-theory can be used to obtain a similar formula for the difference in L-theory associated to any change of Poincaré structures affecting only the linear parts. To see this, let $\mathcal{C}$ be a stable ∞-category and $\Qoppa_0 \to \Qoppa$ a map of Poincaré structures on $\mathcal{C}$ which induces an equivalence on the underlying bilinear parts, which we then denote simply by $\mathrm{B} = \mathrm{B}_{\Qoppa_0} = \mathrm{B}_{\Qoppa}$, and let us write Λ_0 and Λ or the linear parts of $\Qoppa$ and $\Qoppa_0$, respectively. Then the cofiber $\mathrm{Lin} := \mathrm{cof}[\Qoppa_0 \Rightarrow \Qoppa] = \mathrm{cof}[\Lambda_0 \Rightarrow \Lambda]$ is an exact functor on $\mathcal{C}$ and the Poincaré structures $\Qoppa_0$ and $\Qoppa$ share the same underlying duality, which we will just denote by $\mathrm{D} : \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$. We then define the *relative* L*-theory* associated to the map $\Qoppa_0 \Rightarrow \Qoppa$ to be the cofiber

$$\mathrm{L}(\mathcal{C}, \Qoppa_0, \Qoppa) := \mathrm{cof}[\mathrm{L}(\mathcal{C}, \Qoppa_0) \to \mathrm{L}(\mathcal{C}, \Qoppa)].$$

When $\Qoppa_0 = \Qoppa^{\mathrm{q}}_{\mathrm{D}}$ this is just the normal L-theory functor. Corollary 5.0.6 can then be bootstrapped to obtain a formula for the relative L-theory spectrum associated to any map $\Qoppa_0 \Rightarrow \Qoppa$ as above. Let us write $\Theta_0 \Rightarrow \Theta$ for the induced map on co-linear parts of $\Qoppa_0$ and $\Qoppa$, where we recall that the co-linear part of a quadratic functor Ψ is given by the fibre of the map $\Psi(-) \Rightarrow \mathrm{B}_{\Psi}(-,-)^{\mathrm{hC}_2}$.

5.0.10. **Corollary.** *The relative* L*-theory spectrum* $\mathrm{L}(\mathcal{C}, \Omega_0, \Omega)$ *is given by any of the equivalent equalizer formulas below:*

$$\begin{aligned}\mathrm{L}(\mathcal{C}, \Omega_0, \Omega) =& \mathrm{Eq}\left[\tilde{\mathrm{B}}(\mathrm{DLin}, \mathrm{DLin})^{\mathrm{hC}_2} \rightrightarrows \tilde{\Theta}_0^{[1-\sigma]}(\mathrm{DLin})\right] \\ =& \mathrm{Eq}\left[\tilde{\Omega}(\mathrm{DLin}) \rightrightarrows \tilde{\mathrm{B}}^{[1-\sigma]}(\mathrm{DLin}, \mathrm{DLin})\right] \\ =& \mathrm{Eq}\left[\tilde{\mathrm{B}}(\mathrm{DLin}, \mathrm{DLin}) \rightrightarrows \tilde{\Omega}_0^{[1-\sigma]}(\mathrm{DLin})\right] \\ =& \mathrm{Eq}\left[\tilde{\Lambda}(\mathrm{DLin}) \rightrightarrows \tilde{\mathrm{B}}^{[1-\sigma]}(\mathrm{DLin}, \mathrm{DLin})_{\mathrm{hC}_2}\right]\end{aligned}$$

5.0.11. **Example.** Let R be an ordinary ring with involution. For $m \in \mathbb{Z}$ consider the map of Poincaré structures $\Omega_R^{\geq m+1} \to \Omega_R^{\geq m}$, whose cofiber is represented in $\mathcal{D}^{\mathrm{p}}(R)$ by the complex $N := \hat{\mathrm{H}}^m(R)[m]$ consisting of the m' th Tate cohomology group lying in degree m. Using the first and last equalizer formulas for relative L-theory in Corollary 5.0.10 we get

$$\begin{aligned}\mathrm{L}(\mathcal{D}^{\mathrm{p}}(R), \Omega_R^{\geq m+1}, \Omega_R^{\geq m}) =& \mathrm{Eq}\left[(N \otimes_R N)^{\mathrm{hC}_2} \rightrightarrows N \otimes_R \tau_{\leq m}(R^{\mathrm{tC}_2})\right] \\ =& \mathrm{Eq}\left[N \otimes_R \tau_{\geq m}(R^{\mathrm{tC}_2}) \rightrightarrows (\mathbb{S}^{[1-\sigma]} \otimes (N \otimes_R N))_{\mathrm{hC}_2}\right],\end{aligned}$$

where, as in Example 5.0.9 we use the involution on R to switch between left and right actions in forming the tensor product over R. The second equalizer formula then shows that this relative L-spectrum is $(2m-1)$-connective, and if R is Noetherian of finite global dimension d then the first equalizer formula shows that it is also $(2m+d)$-coconnective, in consistency with [CDH+26, Corollary 1.2.8 and Corollary 1.3.7].

Additional applications:

i) Normal L-theory is an étale sheaf and coincides with the 2-completed étale sheafification of symmetric L-theory for finite type $\mathbb{F}_2$-schemes (and some $\mathbb{Z}_2$-schemes) - work in progress with Thomas Nikolaus.
ii) Hermitian Gabber rigidity (Markus Land).
iii) Whiteshift and chromatic purity for L-theory (Jordan Levin).
iv) $\mathbb{A}^1$-invariant GW is insensitive to form flavour.

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