# COHOMOLOGICAL KERNELS OF MIXED EXTENSIONS IN CHARACTERISTIC 2

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ABSTRACT. Let $F$ be a field of characteristic 2, $\Omega_F^m$ the space of $m$-differential forms over $F$, $d:\Omega_F^{m-1}\longrightarrow\Omega_F^m$ the differential operator, and $H_2^{m+1}(F)$ the cokernel of the Artin-Schreier operator $\wp:\Omega_F^m\longrightarrow\Omega_F^m/d\Omega_F^{m-1}$ given on generators by: $\wp(x\frac{dx_1}{x_1}\wedge\cdots\wedge\frac{dx_m}{x_m})=(x^2-x)\frac{dx_1}{x_1}\wedge\cdots\wedge\frac{dx_m}{x_m}+d\Omega_F^{m-1}$. It was shown in [8] that given a multiquadratic purely inseparable extension $L/F$ and a separable quadratic extension $K/F$, then the kernel of the natural homomorphism $H_2^{m+1}(F)\longrightarrow H_2^{m+1}(K\cdot L)$ is equal to the sum of the kernels of the extensions $L/F$ and $K/F$. In this paper we extend this result to the compositum of a finite purely inseparable multiquadratic extension with a separable biquadratic extension, and we prove that this splitting property of kernels is no longer true by computing the kernel of the compositum of a separable quadratic (or biquadratic) extension with a simple purely inseparable extension of arbitrary degree.

## 1. INTRODUCTION

Throughout this paper $F$ denotes a field of characteristic 2 and $W_q(F)$ the Witt group of nonsingular quadratic forms over $F$. Given a field extension $K$ of $F$, one asks for the kernel of the homomorphism $W_q(F)\longrightarrow W_q(K)$ induced by scalar extension. The same type of question arises when considering the groups $I_q^n(F):=I^{n-1}(F)\otimes W_q(F)$ instead of $W_q(F)$ (for $n\geq 1$), where $I^k(F)$ denotes the $k$-th power of the fundamental ideal $I(F)$ of the Witt ring $W(F)$ of regular symmetric bilinear forms over $F$, and $\otimes$ is the module action of $W(F)$ on $W_q(F)$. A very nice general overview on the advances, up until 2015, on this problem can be found in [11, Section 2].

One important tool to study these kernels is a celebrated Theorem of Kato, see [13], which connects quadratic forms in characteristic 2 and differential forms over $F$, this result allows to interpret the above kernels as $H_2^{m+1}(K/F):=\mathrm{Ker}(H_2^{m+1}(F)\longrightarrow H_2^{m+1}(K))$, where $H_2^{m+1}(F)$ is the cokernel of the Artin-Schreier operator $\wp:\Omega_F^m\longrightarrow\Omega_F^m/d\Omega_F^{m-1}$ induced by the Artin-Schreier map $\wp:F\longrightarrow F$, $x\mapsto x^2-x$. Here $\Omega_F^m$ is the space of $m$-differential forms over $F$ and $d:\Omega_F^{m-1}\longrightarrow\Omega_F^m$ is the differential operator. For $x\in\Omega_F^m$, we denote by $\overline{x}$

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the class of $x$ in the quotient group $\Omega_F^m/(\wp(\Omega_F^m)+\mathrm{d}\Omega_F^{m-1})$. Also, for $A \subseteq \Omega_F^m$ we denote by $\overline{A}$ its class in $\Omega_F^m/(\wp(\Omega_F^m)+\mathrm{d}\Omega_F^{m-1})$ (see below for the details).

The kernel $H_2^{m+1}(K/F)$ has been computed in the following cases:

- (A1) $K/F$ is purely transcendental [2, Lem. 2.17].
- (A2) $K/F$ is quadratic [4, Prop. 6.4] and [2, Lem. 2.18].
- (A3) $K/F$ is biquadratic separable [6, Th. 19].
- (A4) $K/F$ is a general quartic extenion [5].
- (A5) $K = F(Q)$ the function field of a projective quadric given by a bilinear Pfister form $Q$ of arbitrary dimension [2, Th. 4.1], or a quadratic Pfister form $Q$ of dimension $2^{n+1}$ such that $m \leq n$ [2, Th. 5.5] and [9, Page 655], or $n = 0, 1$ and $m$ arbitrary [4, Prop. 6.4] and [7, Th. 1.6].
- (A6) $K/F$ is multiquadratic of separability degree $\leq 2$ [8, Prop. 2 and 3].
- (A7) $K$ is the compositum of a multiquadratic extension of $F$ of separability degree $\leq 2$ with the function field of a projective quadric given by a bilinear Pfister form [9].
- (A8) $K/F$ is an arbitrary purely inseparable extension (including the case of infinite degree) [10]. This case was also treated independently by Sobiech in [15].

The field $K$ in the cases (A3), (A6) and (A7) is given as a compositum of some fields extensions of $F$. The remarkable fact in these cases is that $K$ satisfies a *"kernel splitting property"*, meaning that the kernel $H_2^{m+1}(K/F)$ splits into the kernels of the fields composing $K$.

To make the statement *kernel splitting property* more precise we give the following definition.

**Definition 1.1.** *Let $K/F$ be an extension of fields of characteristic $2$. Let $E_1, E_2$ be subfields of $K/F$ such that $K = E_1 \cdot E_2$ is the compositum of $E_1$ and $E_2$, and $E_1 \cap E_2 = F$. We say that $K/F$ satisfies the kernel splitting property with respect to $E_1$ and $E_2$ if for any integer $m \geq 1$, we have the equality*

$$H_2^{m+1}(K/F) = H_2^{m+1}(E_1/F) + H_2^{m+1}(E_2/F).$$

This property is strongly dependent on the subfields as Example 6.2 shows.

The motivation of this paper is to discuss the kernel splitting property for other examples of kernels completing those cited before. More precisely, we will prove that the results of [8, 9] (cited in the cases (A6) and (A7)) extend, in the sense of the kernel splitting property, to any multiquadratic purely inseparable extension composed with a biquadratic separable extension. More precisely, we have the following theorem:

**Theorem 1.2.** *Let $L/F$ be a finite purely inseparable multiquadratic extension and $K/F$ a biquadratic separable extension. Then, the extension $K \cdot L/F$ satisfies the kernel splitting property with respect to $K$ and $L$.*

Recall that for $L = F(\sqrt{b_1}, \ldots, \sqrt{b_s})$ a purely inseparable multiquadratic extension of $F$, and $K = F(\alpha_1, \alpha_2)$ a biquadratic separable extension of $F$ given by $\alpha_i^2 + \alpha_i = a_i \in F$, we have $H_2^{m+1}(L/F) = \sum_{i=1}^{s} \overline{\Omega_F^{m-1} \wedge \mathrm{d}b_i}$ [8, Prop. 2] and $H_2^{m+1}(K/F) = \overline{a_1 \nu_F(m)} + \overline{a_2 \nu_F(m)}$ [6, Th. 19], where $\nu_F(m)$ is the kernel of the Artin-Schreier operator. In particular, Theorem 1.2 gives a complete description of the kernel $H_2^{m+1}(K \cdot L/F)$ for any $m \geq 1$.

It is worth mentioning that the kernel splitting property does not hold in general. For instance a non simple purely inseparable extension of exponent at least 2 does not have this property as it is shown in [10, Theorem 4.1], moreover in this case the elements that compose the kernel can be easily seen to correspond to some particular subfields of the extension, see comment before Theorem 4.1 of [10]. When considering separable extensions, namely a biquadratic separable extension it is shown in [6, Theorem 34] that we also have a kernel splitting property. Now for the case of mixed extensions, say $K \cdot L$ where $K$ is a quadratic or biquadratic separable extension of $F$ and $L$ a purely inseparable extension of $F$ of exponent at least 2 we will show that the kernel $H_2^{m+1}(K \cdot L/F)$ contains elements that do not come neither from the kernel $H_2^{m+1}(K/F)$ nor from the kernel $H_2^{m+1}(L/F)$.

Given a finite purely inseparable extension $L/F$ there is a power of 2, say $2^r$, such that $L^{2^r} \subseteq F$. This behavior somehow extends to differentials contained in $L \cdot \Omega_F^m$ and the next definition arises from the following observations.

Assume that $L = F(\sqrt[2^{n+1}]{b})$ and let $\frac{\mathrm{d}f}{f} \in \nu_L(1) \cap (E \cdot \Omega_F^1)_L$, where $E = F(\sqrt[2^n]{b})$. Set $\beta = \sqrt[2^{n+1}]{b}$ and write $f = f_0 + \beta f_1$ with $f_0, f_1 \in F(\beta^2) = E$. Then $\frac{\mathrm{d}f}{f} = \frac{f_0}{f_0+\beta f_1}\frac{\mathrm{d}f_0}{f_0} + \frac{\beta f_1}{f_0+\beta f_1}\frac{\mathrm{d}f_1}{f_1} + \frac{\beta f_1}{f_0+\beta f_1}\frac{\mathrm{d}\beta}{\beta}$. Since $\mathrm{d}\beta \notin \Omega_F^1$ we conclude that $f = f_0 \in E$. Writing $f = \sum_{j=0}^{2^n-1} h_j \beta^{2j}$ with $h_0, \ldots, h_{2^n-1} \in F$, we see that $\frac{\mathrm{d}f}{f} = \sum_{j=0}^{2^n-1} \frac{h_j \beta^{2j}}{f}\frac{\mathrm{d}h_j}{h_j}$. More generally, considering the generators $\frac{\mathrm{d}f_1}{f_1} \wedge \cdots \wedge \frac{\mathrm{d}f_m}{f_m}$ of the group $\nu_L(m) \cap (E \cdot \Omega_F^m)_L$, we express each $\frac{\mathrm{d}f_i}{f_i}$ as above, multiply them out and squaring several times if necessary (see definition on Section 2), we obtain a differential defined over $F$ modulo powers of exact forms. Bearing this in mind we propose the following definition:

**Definition 1.3.** *Let $L = F(\sqrt[2^{n+1}]{b})$ where $b \in F \setminus F^2$ and $n$ is a positive integer. Let's write $\beta = \sqrt[2^{n+1}]{b}$. For any integer $l \geq 0$, define $G_L^l(m)$ as the subgroup of $\Omega^m_{F(\beta^{2^{l+1}})}$ generated by the differentials:*

$$\sum_{\substack{(k_1,k_2,\ldots,k_m)\\ 0\leq k_s\leq 2^n-1}} \left(\prod_{i=1}^{m} \frac{\beta^{2k_i} f_{i,k_i}}{f_i}\right)^{2^l} \frac{\mathrm{d}f_{1,k_1}}{f_{1,k_1}} \wedge \cdots \wedge \frac{\mathrm{d}f_{m,k_m}}{f_{m,k_m}}$$

*where $f_i = \sum_{k_i=0}^{2^n-1} f_{i,k_i}\beta^{2k_i}$ and $f_{i,k_i} \in F$ for each $1 \leq i \leq m$.*

This definition easily generalizes when the extension $L/F$ is finite purely inseparable.

For the case of separable quadratic extensions, we will prove the following theorem.

**Theorem 1.4.** *Let $K = F(\alpha)$ be a separable quadratic extension of $F$ given by $\alpha^2 + \alpha = a \in F \setminus \wp(F)$, and $L = F(\beta)$ a simple purely inseparable extension of $F$ given by $\beta^{2^{n+1}} = b \in F \setminus F^2$. Suppose that $a \in F^{2^l}$ for some $l > n$. Then, we have*

$$H_2^{m+1}(K \cdot L/F) = \sum_{k=1}^{n} \sum_{j=1}^{2^{k-1}} \overline{b^{2j-1}(\mathrm{d}\Omega_F^{m-1})^{[2^k]}} + \overline{\Omega_F^{m-1} \wedge \mathrm{d}b} + \overline{aG_L^n(m)}.$$

With the same notations as in Theorem 1.4, the hypothesis that $a \in F^{2^l}$ for some $l > n$ is obviously realized without changing the separable extension $K/F$. Moreover, the group

$$\sum_{k=1}^{n} \sum_{j=1}^{2^{k-1}} \overline{b^{2j-1}(\mathrm{d}\Omega_F^{m-1})^{[2^k]}} + \overline{\Omega_F^{m-1} \wedge \mathrm{d}b}$$

is nothing but the kernel of the extension $L/F$ that was considered in [9]. However, the group $\overline{a\nu_F(m)}$, which is the kernel of $K/F$ [4, Prop. 6.4], does not appear explicitly in the theorem, this is because $a\nu_F(m)$ is contained in $aG_L^n(m)$.

It is not difficult to see that Theorem 1.4 can be generalized to consider $L$ a finite purely inseparable extension of $F$.

On a different direction, Theorem 1.4 extends to the case of biquadratic separable extensions as follows:

**Theorem 1.5.** *Let $K = F(\alpha_1, \alpha_2)$ be a biquadratic separable extension of $F$ such that $\alpha_i^2 + \alpha_i = a_i \in F \setminus \wp(F)$ for $1 \leq i \leq 2$, and $L = F(\beta)$ a simple purely inseparable extension of $F$ such that $\beta^{2^{n+1}} = b \in F \setminus F^2$. Suppose that $a_1, a_2 \in F^{2^l}$ for some $l > n$. Then, we have*

$$\begin{aligned} H_2^{m+1}(K \cdot L/F) &= \sum_{k=1}^{n} \sum_{j=1}^{2^{k-1}} \overline{b^{2j-1}(\mathrm{d}\Omega_F^{m-1})^{[2^k]}} + \overline{\Omega_F^{m-1} \wedge \mathrm{d}b} + \\ &+ \overline{a_1 G_L^n(m)} + \overline{a_2 G_L^n(m)}. \end{aligned}$$

The rest of this paper is organized as follows. In section 2 we recall some results on differential forms that we need, like a description of the leading coefficient of an exact differential form (Lemma 2.2), and a useful result due to Kato (Lemma 2.3). In section 3 we give the proof of Theorem 1.4. This proof uses intensively the action of the square operator on the group $\nu_{F(\beta)}(m) \cap (F(\beta^2) \cdot \Omega_F^m)_{F(\beta)}$, where $F(\beta)/F$ is a simple purely inseparable extension (Proposition 3.3), and a precise description of the $F$-component of $2^j$ powers of exact differentials forms (Lemmas 3.5 and 3.6). Section 4 states the preliminary results needed for the proofs of Theorems 1.2 and 1.5, these proofs use a very different technique when compare to the one used in Theorem 1.4. Specifically, for Theorem 1.4 we will work over the separable quadratic extension $K = F(\alpha)$, $\alpha^2 + \alpha = a \in F \setminus \wp(F)$, and use the description of the groups $\wp(\Omega_K^m)$ and $\mathrm{d}\Omega_K^{m-1}$ in terms of the groups $\wp(\Omega_F^m)$ and $\mathrm{d}\Omega_F^{m-1}$. However, for Theorem 1.5 we will first work over the simple purely inseparable extension $L = F(\beta)$, so that we apply the kernel of the biquadratic separable extension $K = F(\alpha_1, \alpha_2)$ found in [6]. This allows us to work with a relation of the shape

$$w_L \in a_1\nu_L(m) + a_2\nu_L(m) + \wp(\Omega_L^m) + \mathrm{d}\Omega_L^{m-1}, \text{ where } w \in \Omega_F^m.$$

Subsequently using Proposition 4.4, we descend the above relation to one defined over the field $L^{(2)} := F(\beta^2)$, that is, for the same $w$ we reduce to a relation

$$w_L \in a_1(\nu_{L^{(2)}}(m))_L + a_2(\nu_{L^{(2)}}(m))_L + (\wp(\Omega_{L^{(2)}}^m) + \mathrm{d}\Omega_{L^{(2)}}^{m-1})_L.$$

Based on this last relation, we prove Theorem 1.2 and conclude the proof of Theorem 1.5 using arguments similar as those in the proof of Theorem 1.4 (Section 5). Note that Proposition 4.4 uses Lemma 4.3 which is based on some important exact sequences concerning Izhboldin's groups done by the first author and B. Jacob in [5]. We finish with Section 6 giving an example that illustrates the strong dependency of the kernel splitting property on the subfields defining the compositum.

## 2. Differential forms

2.1. **Background on differential forms.** For any integer $m \geq 1$, let $\Omega_F^m = \wedge^m \Omega_F^1$ be the space of $m$-differential forms over $F$, where $\Omega_F^1$ is the $F$-vector space generated by the symbols $\mathrm{d}x$, $x \in F$, subject to the relations: $\mathrm{d}(x+y) = \mathrm{d}x + \mathrm{d}y$ and $\mathrm{d}(xy) = x\mathrm{d}y + y\mathrm{d}x$ for all $x, y \in F$ (we take $\Omega_F^0 = F$). In particular, there is an $F^2$-linear map $F \longrightarrow \Omega_F^1$, given by $x \mapsto \mathrm{d}x$. This map extends to the differential operator $\mathrm{d} : \Omega_F^m \longrightarrow \Omega_F^{m+1}$ as follows: $\mathrm{d}(x\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_m) = \mathrm{d}x \wedge \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_m$.

Let $\mathcal{B} = \{e_i \mid i \in I\}$ be a 2-basis of $F$, that is, the set

$$\left\{ \prod_{i \in I} e_i^{\epsilon_i} \mid \epsilon_i \in \{0,1\}, \text{ and } \epsilon_i = 0 \text{ for almost all } i \in I \right\}$$

is an $F^2$-basis of $F$. We fix an ordering on $I$ and consider $\sum_{m,F} := \{\sigma : \{1,\ldots,m\} \longrightarrow I \mid \sigma(i) < \sigma(j) \text{ whenever } i < j\}$, this set is then equipped with the lexicographic ordering, thus the set $\{\frac{\mathrm{d}e_\gamma}{e_\gamma} \mid \gamma \in \sum_{m,F}\}$ is an $F$-basis of $\Omega_F^m$, where $\frac{\mathrm{d}e_\gamma}{e_\gamma} = \frac{\mathrm{d}e_{\gamma(1)}}{e_{\gamma(1)}} \wedge \cdots \wedge \frac{\mathrm{d}e_{\gamma(m)}}{e_{\gamma(m)}}$. Also for any $i \in I$, we define $F_i$ as the field $F^2(e_j \mid j \leq i)$ and $F_{<i}$ as $F^2(e_j \mid j < i)$. Similarly, for $\gamma \in \sum_{m,F}$ let $\Omega_{F,\gamma}^m$ be the subspace of $\Omega_F^m$ generated by $\frac{\mathrm{d}e_\sigma}{e_\sigma}$ with $\sigma \leq \gamma$ and $\Omega_{F,<\gamma}^m$ the subspace of $\Omega_F^m$ generated by $\frac{\mathrm{d}e_\sigma}{e_\sigma}$ with $\sigma < \gamma$. Thus, there is a filtration of $\Omega_F^m$ given by $\Omega_{F,\gamma}^m$, $\gamma \in \sum_{m,F}$ (and $\Omega_{F,<\gamma}^m$, $\gamma \in \sum_{m,F}$).

The maximal multi-index of a nonzero $u \in \Omega_F^m$ will be denoted by $max(u)$.

The well known Artin-Schreier map $\wp : F \longrightarrow F$, defined by $\wp(a) = a^2 - a$ for $a \in F$, extends to $\Omega_F^m$ by setting $\wp : \Omega_F^m \longrightarrow \Omega_F^m/\mathrm{d}\Omega_F^{m-1}$ as $\wp(\sum_{\sigma\leq\gamma} c_\sigma \frac{\mathrm{d}e_\sigma}{e_\sigma}) = \sum_{\sigma\leq\gamma}(c_\sigma^2 - c_\sigma)\frac{\mathrm{d}e_\sigma}{e_\sigma} + \mathrm{d}\Omega_F^{m-1}$ and it is also called the Artin-Schreier operator. Its definition depends on the 2-basis but is independent of that modulo $\mathrm{d}\Omega_F^{m-1}$. The kernel of the Artin-Schreier operator is denoted by $\nu_F(m)$. Also $H_2^{m+1}(F)$ is defined as the cokernel of $\wp$ and we may consider the group $H_2^{m+1}(F)$ as the quotient $\Omega_F^m/(\wp(\Omega_F^m) + \mathrm{d}\Omega_F^{m-1})$.

The square operator defined by: $w = \sum_{\sigma\leq\gamma} c_\sigma \frac{\mathrm{d}e_\sigma}{e_\sigma} \mapsto w^{[2]} = \sum_{\sigma\leq\gamma} c_\sigma^2 \frac{\mathrm{d}e_\sigma}{e_\sigma}$ depends on the choice of the 2-basis, but it is well defined modulo the subgroup $\mathrm{d}\Omega_F^{m-1}$ and thus is a group homomorphism.

For any $w = \sum_{\sigma\leq\gamma} c_\sigma \frac{\mathrm{d}e_\sigma}{e_\sigma} \in \Omega_F^m$ we write $w^{[2]} = \sum_{\sigma\leq\gamma} c_\sigma^2 \frac{\mathrm{d}e_\sigma}{e_\sigma} + \mathrm{d}s_1$ for some $s_1 \in \Omega_F^{m-1}$. Using this expression and applying the square operator again we get $w^{[2^2]} := (w^{[2]})^{[2]} = \sum_{\sigma\leq\gamma} c_\sigma^4 \frac{\mathrm{d}e_\sigma}{e_\sigma} + (\mathrm{d}s_1)^{[2]} + \mathrm{d}s_2$ for some $s_2 \in \Omega_F^{m-1}$, and thus $w^{[2^2]} = \sum_{\sigma\leq\gamma} c_\sigma^4 \frac{\mathrm{d}e_\sigma}{e_\sigma} + \wp(\mathrm{d}s_1) + \mathrm{d}(s_1 + s_2)$. More generally, we have $w^{[2^k]} = \sum_{\sigma\leq\gamma} c_\sigma^{2^k} \frac{\mathrm{d}e_\sigma}{e_\sigma} + \wp(\mathrm{d}s) + \mathrm{d}t$ for some $s, t \in \Omega_F^{m-1}$, and therefore $\overline{w^{[2^k]}} = \overline{\sum_{\sigma\leq\gamma} c_\sigma^{2^k} \frac{\mathrm{d}e_\sigma}{e_\sigma}}$ in $H_2^{m+1}(F)$.

2.2. **Tools for differential forms.** In this part we gather some results about differential forms.

Some of our proofs will consider changing the initial 2-basis by another one without changing the filtrations, namely $(\Omega_{F,\gamma}^m)_{\gamma\in\Sigma_{m,F}}$ and $(\Omega_{F,<\gamma}^m)_{\gamma\in\Sigma_{m,F}}$ (*resp.* $(\nu_{F,\alpha}(m))_{\gamma\in\Sigma_{m,F}}$ and $(\nu_{F,<\gamma}(m))_{\gamma\in\Sigma_{m,F}}$) given by the original 2-basis. One instance of this behavior is given by the following proposition.

**Proposition 2.1.** *Let $\mathcal{B} = \{e_i \mid i \in I\}$ be a 2-basis of $F$. We choose an ordering on $I$ and let $\sigma \in \Sigma_{m,F}$. Let $f_{\sigma(i)} = r_i + s_i e_{\sigma(i)}$ be such that $r_i, s_i \in F_{<\sigma(i)}$ and $s_i \neq 0$ for all $1 \leq i \leq m$, i.e., $f_{\sigma(i)} \in F_{\sigma(i)} \setminus F_{<\sigma(i)}$. We introduce the set $\mathcal{B}' = \{e'_i \mid i \in I\}$ given as follows:*

$$e'_k = \begin{cases} e_k & \text{if } k \neq \sigma(1), \dots, \sigma(m), \\ f_k & \text{if } k \in \{\sigma(1), \dots, \sigma(m)\}, \end{cases}$$

*and we keep the same ordering on $I$. Then:*
*(1) $\mathcal{B}'$ is a 2-basis of $F$.*
*(2) $F^2(e_j \mid j < \sigma(i)) = F^2(e'_j \mid j < \sigma(i))$ and $F^2(e_j \mid j \leq \sigma(i)) = F^2(e'_j \mid j \leq \sigma(i))$.*
*(3) For any $w \in \Omega_F^m$, the maximal multi-indices of $w$ with respect to $\mathcal{B}$ and $\mathcal{B}'$ are the same.*
*(4) The group $\Omega_{F,\sigma}^m$ is the same for $\mathcal{B}$ and $\mathcal{B}'$. Similarly for $\Omega_{F,<\sigma}^m$ and $\nu_{F,<\sigma}(m)$.*

*Proof.* (1) and (2) are straight forward.

(3) Let $w \in \Omega_F^m$ and $\tau \in \Sigma_{m,F}$ be the maximal multi-index of $w$ with respect to $\mathcal{B}$. We will prove that $\tau$ is also the maximal multi-index of $w$ with respect to $\mathcal{B}'$. It suffices to prove the statement when $w = \frac{\mathrm{d}e_{\tau(1)}}{e_{\tau(1)}} \wedge \cdots \wedge \frac{\mathrm{d}e_{\tau(m)}}{e_{\tau(m)}}$. Let $\sigma \cap \tau$ be the intersection of the multi-indices $\sigma$ and $\tau$ defined as the set $\{i \in I \mid \sigma(k) = i = \tau(l) \text{ for some } 1 \leq k, l \leq m\}$.

• Suppose $\sigma \cap \tau = \emptyset$. Then, $w = \frac{\mathrm{d}e'_{\tau(1)}}{e'_{\tau(1)}} \wedge \cdots \wedge \frac{\mathrm{d}e'_{\tau(m)}}{e'_{\tau(m)}}$, and thus $\tau$ is the maximal multi-index of $w$ with respect to $\mathcal{B}'$.

• Suppose $\sigma \cap \tau \neq \emptyset$. Let $s$ be the cardinal of $\sigma \cap \tau$, and let $k_1 < k_2 < \cdots < k_s$ and $l_1 < l_2 < \cdots < l_s$ be such that $\sigma(k_i) = \tau(l_i)$ for $1 \leq i \leq s$. Let $w' = \wedge_{\substack{i \neq l_1, \cdots, l_s \\ 1 \leq i \leq m}} \frac{\mathrm{d}e_{\tau(i)}}{e_{\tau(i)}}$. Clearly, $w' = \wedge_{\substack{i \neq l_1, \cdots, l_s \\ 1 \leq i \leq m}} \frac{\mathrm{d}e'_{\tau(i)}}{e'_{\tau(i)}}$ and,

$$w = w' \wedge \frac{\mathrm{d}e_{\tau(l_1)}}{e_{\tau(l_1)}} \wedge \cdots \wedge \frac{\mathrm{d}e_{\tau(l_s)}}{e_{\tau(l_s)}}, \tag{1}$$

and

$$w' \wedge \frac{\mathrm{d}f_{\sigma(k_1)}}{f_{\sigma(k_1)}} \wedge \cdots \wedge \frac{\mathrm{d}f_{\sigma(k_s)}}{f_{\sigma(k_s)}} = \frac{\mathrm{d}e'_\tau}{e'_\tau} \tag{2}$$

because $\sigma(k_i) = \tau(l_i)$ and $f_{\sigma(k_i)} = e'_{\sigma(k_i)}$.

Moreover, $e_{\sigma(k_i)} = s_{k_i}^{-1}(r_{k_i} + f_{\sigma(k_i)})$ with $r_{k_i}, s_{k_i} \in F^2(e_j \mid j < \sigma(k_i))$, and by (2) we see that $r_{k_i}, s_{k_i} \in F^2(e'_j \mid j < \sigma(k_i)) = F^2(e'_j \mid j < \tau(l_i))$.

Substituting the expression for $e_{\sigma(k_i)}$ in (1), and using (2), we conclude that $w = c\frac{de'_\tau}{e'_\tau} + w'$, for suitable $c \in F^*$ and $w' \in \Omega_F^m$ whose maximal multi-index with respect to $\mathcal{B}'$ is smaller than $\tau$, i.e., the maximal multi-index of $w$ with respect to $\mathcal{B}'$ is $\tau$.

(4) This is a direct consequence of (3). □

The following Lemma, due to Aravire and Baeza, gives a precise description of the leading coefficient of an exact differential form.

**Lemma 2.2.** *(*[3, Lemma 3.2]*) Let $\mathcal{B} = \{e_i \mid i \in I\}$ be a 2-basis of $F$, and $u = \sum_{\tau \leq \gamma} c_\tau \frac{de_\tau}{e_\tau} \in \mathrm{d}\Omega_F^{m-1}$ ($m \geq 1$) such that $c_\gamma \neq 0$. Then, $c_\gamma = \sum_{i=1}^m x_i e_{\gamma(i)}$, where $x_i \in F_{<\gamma(i)}$ for each $1 \leq i \leq m$.*

The following result, due to Kato, plays a relevant role in the proof of the following Corollary 2.4 and Lemma 2.5.

**Lemma 2.3.** *(*[14, Lemma 2]*) Let $\mathcal{B} = \{e_i \mid i \in I\}$ be a 2-basis of $F$. Let $\gamma \in \sum_{m,F}$ and $x \in F$ be such that $\wp(x\frac{de_\gamma}{e_\gamma}) \in \Omega_{F,<\gamma}^m + \mathrm{d}\Omega_F^{m-1}$. Then, there exist $v \in \Omega_{F,<\gamma}^m$ and $c_i \in F_{\gamma(i)}$, $1 \leq i \leq m$, such that*

$$x\frac{de_\gamma}{e_\gamma} = \frac{dc_1}{c_1} \wedge \cdots \wedge \frac{dc_m}{c_m} + v.$$

By an induction argument on multi-indices and using the above Lemma 2.3 one can prove the following.

**Corollary 2.4.** *Let $\mathcal{B} = \{e_i \mid i \in I\}$ be a 2-basis of $F$, $\gamma \in \Sigma_{m,F}$ and $u \in \nu_F(m)$ nonzero such that $\gamma = max(u)$. Then, $u = \sum_{\sigma \leq \gamma} \frac{df_{\sigma(1)}}{f_{\sigma(1)}} \wedge \cdots \wedge \frac{df_{\sigma(m)}}{f_{\sigma(m)}}$ such that $f_{\sigma(i)} \in F_{\sigma(i)}$ for each $1 \leq i \leq m$.*

We will refer to this result as *Kato's decomposition of elements* in $\nu_F(n)$.

The next Lemma is another result that depends on Lemma 2.3, we ommit its proof.

**Lemma 2.5.** *(*[9, Lemma 3.5]*) Let $\mathcal{B}$ be a 2-basis of $F$, $\gamma \in \Sigma_{m,F}$ and $u, w \in \Omega_F^m$, $v \in \Omega_F^{m-1}$ satisfying $max(w) = \gamma$ and $w = \wp(u) + \mathrm{d}v$. Then, there exist $\xi \in \nu_F(m)$, $u' \in \Omega_{F,<\gamma}^m$ and $v' \in \Omega_F^{m-1}$ such that:*
*(1) $u = \xi + u'$.*
*(2) $w = \wp(u') + \mathrm{d}v'$ and $max(\mathrm{d}v') \leq \gamma$.*

The following result can be found in [9] and we omit the proof.

**Lemma 2.6.** *(*[9, Lemma 3.4]*) Let $\{e_i \mid i \in I\}$ be a 2-basis of an extension $F'$ of $F$, $\gamma \in \Sigma_{m,F'}$ and $f_{\gamma(i)} \in F'_{\gamma(i)}$ for $1 \leq i \leq m$. Let $M_1, \ldots, M_m \in F'$ be such that $M_i \in F'_{<\gamma(i)}$ for $1 \leq i \leq m$. Then:*
*(1) $\left(\sum_{i=1}^m M_i f_{\gamma(i)}\right) \frac{\mathrm{d}f_{\gamma(1)}}{f_{\gamma(1)}} \wedge \cdots \wedge \frac{\mathrm{d}f_{\gamma(m)}}{f_{\gamma(m)}} \in \mathrm{d}\Omega_{F'}^{m-1} + \Omega_{F',<\gamma}^m$.*
*(2) $\frac{\mathrm{d}f_{\gamma(1)}}{f_{\gamma(1)}} \wedge \cdots \wedge \frac{\mathrm{d}f_{\gamma(m)}}{f_{\gamma(m)}} = \frac{\mathrm{d}(M_1 f_{\gamma(1)})}{M_1 f_{\gamma(1)}} \wedge \cdots \wedge \frac{\mathrm{d}(M_m f_{\gamma(m)})}{M_m f_{\gamma(m)}} + \mu$, where $\mu \in \nu_{F',<\gamma}(m)$.*
*(3) If moreover $f_{\gamma(i)}$ and $M_i$ belong to $F, 1 \leq i \leq m$, then combining (1) and (2) we get:*
*(a) $\left(\sum_{i=1}^m M_i f_{\gamma(i)}\right) \frac{\mathrm{d}f_{\gamma(1)}}{f_{\gamma(1)}} \wedge \cdots \wedge \frac{\mathrm{d}f_{\gamma(m)}}{f_{\gamma(m)}} \in (\mathrm{d}\Omega_F^{m-1})_{F'} + (\Omega_F^m)_{F',<\gamma}$.*
*(b) $\mu \in (\nu_F(m))_{F',<\gamma}$.*

If $L$ is a field extension of $F$, then the images of the natural homomorphisms $\Omega_F^m \longrightarrow \Omega_L^m$ and $\nu_F(m) \longrightarrow \nu_L(m)$ are denoted by $(\Omega_F^m)_L$ and $(\nu_F(m))_L$, respectively.

For each $\gamma \in \Sigma_{m,L}$, we denote by $(\Omega_F^m)_{L,<\gamma}$ and $(\nu_F(m))_{L,<\gamma}$ the groups $(\Omega_F^m)_L \cap \Omega_{L,<\gamma}^m$ and $(\nu_F(m))_L \cap \Omega_{L,<\gamma}^m$, respectively.

## 3. Proof of Theorem 1.4

Throughout this section, let $K = F(\alpha)$ and $L = F(\beta)$, where $\alpha^2 + \alpha = a \in F \setminus \wp(F)$ and $\beta^{2^{n+1}} = b \in F \setminus F^2$. We consider $\mathcal{B} = \{e_1 = b, e_2, \ldots\}$ a 2-basis of $F$ together with an ordering. The set $\mathcal{C} = \{e_2, \ldots\} \cup \{\beta\}$ is then a 2-basis of $F(\beta)$ and we consider an ordering on $\mathcal{C}$ so that $\beta$ is its last element.

We begin by showing that the groups $\overline{aG_L^n(m)}$, see Definition 1.3, are part of the kernels we are dealing with.

**Lemma 3.1.** *Let $K$ and $L$ be as above and suppose that $a \in F^{2^n}$. Then, $\overline{aG_L^n(m)} \subset H_2^{m+1}(K \cdot L/F)$.*

*Proof.* Let $\xi = a \sum_{\substack{(k_1,k_2,\ldots,k_m)\\ 0\leq k_s\leq 2^n-1}} \left(\prod_{i=1}^m \frac{\beta^{2k_i} f_{i,k_i}}{f_i}\right)^{2^n} \frac{\mathrm{d}f_{1,k_1}}{f_{1,k_1}} \wedge \cdots \wedge \frac{\mathrm{d}f_{m,k_m}}{f_{m,k_m}}$ be a generator of the group $aG_L^n(m)$, where $f_i = \sum_{k_i=0}^{2^n-1} f_{i,k_i}\beta^{2k_i}$ and $f_{i,k_i} \in F$ for each $1 \leq i \leq m$.

Since $a \in F^{2^n}$, we may write

$$\xi = \sum_{\substack{(k_1,k_2,\ldots,k_m)\\ 0\leq k_s\leq 2^n-1}} \left(\sqrt[2^n]{a}\prod_{i=1}^m \frac{\beta^{2k_i} f_{i,k_i}}{f_i}\right)^{2^n} \frac{\mathrm{d}f_{1,k_1}}{f_{1,k_1}} \wedge \cdots \wedge \frac{\mathrm{d}f_{m,k_m}}{f_{m,k_m}}.$$

Clearly, modulo the group $\wp(\Omega_L^m) + \mathrm{d}\Omega_L^{m-1}$, we have

$$\begin{aligned} \xi &= \sum_{\substack{(k_1,k_2,\ldots,k_m)\\ 0\leq k_s\leq 2^n-1}} \left(\sqrt[2^n]{a}\prod_{i=1}^m \frac{\beta^{2k_i} f_{i,k_i}}{f_i}\right) \frac{\mathrm{d}f_{1,k_1}}{f_{1,k_1}} \wedge \cdots \wedge \frac{\mathrm{d}f_{m,k_m}}{f_{m,k_m}} \\ &= \sqrt[2^n]{a}\,\frac{\mathrm{d}f_1}{f_1} \wedge \cdots \wedge \frac{\mathrm{d}f_m}{f_m} = a\,\frac{\mathrm{d}f_1}{f_1} \wedge \cdots \wedge \frac{\mathrm{d}f_m}{f_m}. \end{aligned}$$

Since $\overline{a\,\frac{\mathrm{d}f_1}{f_1} \wedge \cdots \wedge \frac{\mathrm{d}f_m}{f_m}} \in H_2^{m+1}(K \cdot L/F)$, it follows that $\overline{\xi} \in H_2^{m+1}(K \cdot L/F)$. □

A key point in the proof of Theorem 1.4 is provided by the following observation.

**Lemma 3.2.** *Keeping the same notations as before. For any integer $m \geq 1$ it holds*

$$\nu_{F(\beta)}(m) \cap (F(\beta^2) \cdot \Omega_F^m)_{F(\beta)} = (\nu_{F(\beta^2)}(m))_{F(\beta)}.$$

*Proof.* Since $\mathrm{d}\beta^2 = 0 \in \Omega^1_{F(\beta)}$ and $\Omega^m_{F(\beta^2)} = F(\beta^2)\cdot\Omega_F^m + F(\beta^2)\cdot\Omega_F^{m-1}\wedge \mathrm{d}\beta^2$, it follows that $(\nu_{F(\beta^2)}(m))_{F(\beta)} \subset \nu_{F(\beta)}(m) \cap (F(\beta^2)\cdot\Omega_F^m)_{F(\beta)}$. Conversely, let $w \in \nu_{F(\beta)}(m) \cap (F(\beta^2)\cdot\Omega_F^m)_{F(\beta)}$ be nonzero. Since $w \in F(\beta^2)\cdot\Omega_F^m$, it follows from Corollary 2.4 and the ordering on $\mathcal{C}$ that $w = \sum_{\sigma\leq\gamma} \frac{\mathrm{d}f_{\sigma(1)}}{f_{\sigma(1)}} \wedge \cdots \wedge \frac{\mathrm{d}f_{\sigma(m)}}{f_{\sigma(m)}}$, where $\gamma = max(w)$ and $f_{\sigma(i)} \in L_{\sigma(i)} = L^2(e_j \in \mathcal{B} \mid j \leq \sigma(i))$ for all $1 \leq i \leq m$. Since $L_{\sigma(i)} \subset F(\beta^2)$, we conclude that $w \in (\nu_{F(\beta^2)}(m))_{F(\beta)}$. □

The following proposition gives a description on how the square operator acts on the group $\nu_{F(\beta)}(m) \cap (F(\beta^2)\cdot\Omega_F^m)_{F(\beta)}$.

**Proposition 3.3.** *Keep the same notations as before. Let $\eta \in \nu_{F(\beta)}(m) \cap (F(\beta^2).\Omega_F^m)_{F(\beta)}$. Then, for any positive integer $l$ there are exact differential forms $\mathrm{d}s_i \in F(\beta^2)\Omega_F^m$ and $\widetilde{\eta} \in G_L^l(m)$ such that*

$$\eta = \widetilde{\eta} + \sum_{j=0}^{l-1} (\mathrm{d}s_j)^{[2^j]}. \quad (3)$$

*Proof.* By Lemma 3.2, we have $\eta \in (\nu_{F(\beta^2)}(m))_{F(\beta)}$. It suffices to prove the proposition for the case $\eta = \frac{\mathrm{d}f_1}{f_1} \wedge \cdots \wedge \frac{\mathrm{d}f_m}{f_m}$ with $f_1, \ldots, f_m \in F(\beta^2)$.

Writing $f_i = \sum_{k_i=0}^{2^n-1} f_{i,k_i}\beta^{2k_i}$ for $f_{i,k_i} \in F$, and arguing as we did before (Definition 1.3), it becomes clear that

$$\eta = \sum_{\substack{(k_1,k_2,\ldots,k_m)\\ 0\le k_s \le 2^n-1}} \left(\prod_{i=1}^{m} \frac{\beta^{2k_i} f_{i,k_i}}{f_i}\right) \frac{\mathrm{d}f_{1,k_1}}{f_{1,k_1}} \wedge \cdots \wedge \frac{\mathrm{d}f_{m,k_m}}{f_{m,k_m}}.$$

It is well known that the elements of $\nu_{F(\beta)}(m)$ are invariant under the square operator modulo exact differential forms, then $\eta \equiv \eta^{[2]} \pmod{\mathrm{d}\Omega_{F(\beta)}^{m-1}}$, where

$$\eta^{[2]} \equiv \sum_{\substack{(k_1,k_2,\ldots,k_m)\\ 0\le k_s \le 2^n-1}} \left(\prod_{i=1}^{m} \frac{\beta^{2k_i} f_{i,k_i}}{f_i}\right)^2 \frac{\mathrm{d}f_{1,k_1}}{f_{1,k_1}} \wedge \cdots \wedge \frac{\mathrm{d}f_{m,k_m}}{f_{m,k_m}} \pmod{\mathrm{d}\Omega_{F(\beta)}^{m-1}}.$$

Hence, there is $\mathrm{d}s \in \mathrm{d}\Omega_{F(\beta)}^{m-1}$ such that

$$\eta = \sum_{\substack{(k_1,k_2,\ldots,k_m)\\ 0\le k_s \le 2^n-1}} \left(\prod_{i=1}^{m} \frac{\beta^{2k_i} f_{i,k_i}}{f_i}\right)^2 \frac{\mathrm{d}f_{1,k_1}}{f_{1,k_1}} \wedge \cdots \wedge \frac{\mathrm{d}f_{m,k_m}}{f_{m,k_m}} + \mathrm{d}s,$$

also notice that $\mathrm{d}s \in (F(\beta^2) \cdot \Omega_F^m)_{F(\beta)}$. Applying the square operator again, we have

$$\eta = \sum_{\substack{(k_1,k_2,\ldots,k_m)\\ 0\le k_s \le 2^n-1}} \left(\prod_{i=1}^{m} \frac{\beta^{2k_i} f_{i,k_i}}{f_i}\right)^{2^2} \frac{\mathrm{d}f_{1,k_1}}{f_{1,k_1}} \wedge \cdots \wedge \frac{\mathrm{d}f_{m,k_m}}{f_{m,k_m}} + (\mathrm{d}s)^{[2]} + \mathrm{d}t,$$

for a suitable $\mathrm{d}t \in \mathrm{d}\Omega_{F(\beta)}^{m-1}$, also notice (as above) that $\mathrm{d}t \in (F(\beta^2) \cdot \Omega_F^m)_{F(\beta)}$.

An induction argument shows that for any positive integer $l$ there are exact differentials $\mathrm{d}s_0, \ldots, \mathrm{d}s_{l-1} \in (F(\beta^2) \cdot \Omega_F^m)_{F(\beta)}$ such that

$$\eta = \widetilde{\eta} + \sum_{j=0}^{l-1} (\mathrm{d}s_j)^{[2^j]},$$

where $\widetilde{\eta} = \sum\limits_{\substack{(k_1,k_2,\ldots,k_m)\\ 0\leq k_s\leq 2^n-1}} \left(\prod_{i=1}^{m} \frac{\beta^{2k_i} f_{i,k_i}}{f_i}\right)^{2^l} \frac{\mathrm{d}f_{1,k_1}}{f_{1,k_1}} \wedge \cdots \wedge \frac{\mathrm{d}f_{m,k_m}}{f_{m,k_m}} \in G_L^l(m)$. □

In order to prove our Theorem we need a description of the $F$-component of $\sum_{j=0}^{l-1}(\mathrm{d}s_j)^{[2^j]}$ appearing in equation (3), where $\mathrm{d}s_0, \ldots, \mathrm{d}s_{l-1} \in \mathrm{d}\Omega^{m-1}_{F(\beta)} \cap (F(\beta^2)\cdot\Omega^m_F)_{F(\beta)}$ (Lemmas 3.5 and 3.6).

One can write $\Omega^m_{F(\beta)} = \left(\bigoplus_{i=0}^{2^{n+1}-1} \beta^i \Omega^m_F\right) \oplus \left(\bigoplus_{i=0}^{2^{n+1}-1} \beta^i \Omega^{m-1}_F \wedge \mathrm{d}\beta\right)$, thus every $s_j \in \Omega^m_{F(\beta)}$ can be written as $s_j = \sum_{i=0}^{2^{n+1}-1} \beta^i \xi_{j,i} + \sum_{i=0}^{2^{n+1}-1} \beta^i \chi_{j,i} \wedge \mathrm{d}\beta$ with $\xi_{j,i} \in \Omega^{m-1}_F$ and $\chi_{j,i} \in \Omega^{m-2}_F$ for all $0 \leq i \leq 2^{n+1}-1$ and $0 \leq j \leq n-1$.

Therefore, for each $j$, $\mathrm{d}s_j = \mathrm{d}\xi_{j,0} + \sum_{i=1}^{2^{n+1}-1} \beta^i \mathrm{d}\xi_{j,i} + Z \wedge \mathrm{d}\beta$. Since $\mathrm{d}s_j \in F(\beta^2)\cdot\Omega^m_F$, we see that

$$\mathrm{d}s_j = \mathrm{d}\xi_{j,0} + \sum_{i=1}^{2^n-1} \beta^{2i} \mathrm{d}\xi_{j,2i}\,,\ \xi_{j,2i} \in \Omega^{m-1}_F, 1 \leq i \leq 2^n - 1. \tag{4}$$

As an immediate consequence we have the following:

**Corollary 3.4.** $\mathrm{d}\Omega^{m-1}_{F(\beta^2)} = \mathrm{d}\Omega^{m-1}_{F(\beta)} \cap (F(\beta^2)\cdot\Omega^m_F)_{F(\beta)}$.

To describe the $F$-component of $\sum_{j=0}^{l-1}(\mathrm{d}s_j)^{[2^j]}$ from (3) we use the following lemma.

**Lemma 3.5.** *Let $l$ be an arbitrary positive integer, and $\mathrm{d}s_0, \ldots, \mathrm{d}s_l \in \mathrm{d}\Omega^{m-1}_{F(\beta)} \cap (F(\beta^2)\cdot\Omega^m_F)_{F(\beta)}$. Then, the $F$-component of $\sum_{j=0}^{l}(\mathrm{d}s_j)^{[2^j]}$ is contained in the group*

$$\sum_{j=0}^{l} \sum_{\substack{i2^{j-n}=t\in\mathbb{N}\\ 0\leq i\leq 2^n-1}} b^t (\mathrm{d}\Omega^{m-1}_F)^{[2^j]}.$$

*Proof.* By equation (4), using the fact that $[2^j]$ is a homomorphism, we have

$$\sum_{j=0}^{l}(\mathrm{d}s_j)^{[2^j]} = \sum_{j=0}^{l}(\mathrm{d}\xi_{j,0})^{[2^j]} + \sum_{j=0}^{l}\sum_{i=1}^{2^n-1} \beta^{i2^{j+1}} (\mathrm{d}\xi_{j,2i})^{[2^j]}, \tag{5}$$

where $\xi_{j,2i} \in \Omega^{m-1}_F$ for all $0 \leq i \leq 2^n - 1$. We are only interested in the part that is defined over $F$, thus we just need to consider the indices $i$ and $j$ such that $\beta^{i2^{j+1}} = b^t$ for some non-negative integer $t$. This is when $i2^{j+1} = 2^{n+1}t$, that is when $i2^{j-n} \in \mathbb{N}$ with $0 \leq i \leq 2^n - 1$. □

**Lemma 3.6.** *For any integer $0 \le l \le n$, the group $\sum_{j=0}^{l} \sum_{\substack{i2^{j-n}=t\in\mathbb{N} \\ 0\le i\le 2^n-1}} b^t(\mathrm{d}\Omega_F^{m-1})^{[2^j]}$ is contained in the group*

$$\sum_{k=1}^{n}\sum_{j=1}^{2^{k-1}} b^{2j-1}(\mathrm{d}\Omega_F^{m-1})^{[2^k]} + \Omega_F^{m-1}\wedge \mathrm{d}b + \wp(\Omega_F^m) + \mathrm{d}\Omega_F^{m-1}.$$

*Proof.* Let $A = b^t(\mathrm{d}\lambda)^{[2^j]}$ such that $\lambda \in \Omega_F^{m-1}$, $i2^{j-n} = t \in \mathbb{N}$, $0 \le j \le l \le n$ and $0 \le i \le 2^n - 1$. We have to prove that $A$ belongs to

$$\sum_{k=1}^{n}\sum_{j=1}^{2^{k-1}} b^{2j-1}(\mathrm{d}\Omega_F^{m-1})^{[2^k]} + \Omega_F^{m-1}\wedge \mathrm{d}b + \wp(\Omega_F^m) + \mathrm{d}\Omega_F^{m-1}.$$

(i) Suppose $j = 0$. Then $i = 0$ because $i2^{j-n} = t \in \mathbb{N}$ and $0 \le i \le 2^n - 1$. Therefore $A = \mathrm{d}\lambda$.

(ii) Suppose $j \ge 1$. If $t$ is even, then $t = 2t'$ where $t' = i2^{j-n-1}$. Since $A \equiv b^{t'}(\mathrm{d}\lambda)^{[2^{j-1}]} \pmod{\wp(\Omega_F^m)+\mathrm{d}\Omega_F^{m-1}}$, we reduce to the element $b^{t'}(\mathrm{d}\lambda)^{[2^{j-1}]}$. Repeating this argument, if necessary, we may suppose that $t$ is odd.

Let $t = 2k - 1$ for some integer $k$, since $0 \le i \le 2^n - 1$ we have $0 \le t < 2^j$, therefore $1 \le k \le 2^{j-1}$.

Then, we proved that, modulo $\wp(\Omega_F^m) + \mathrm{d}\Omega_F^{m-1}$, the element $A$ belongs to $b^{2k-1}(\mathrm{d}\Omega_F^{m-1})^{[2^j]}$, where $1 \le k \le 2^{j-1}$ and $1 \le j \le n$. □

The kernel of a simple purely inseparable extension is computed in [10, Theorem 3.1], and [15, Corollary 3.15] for $p = 2$. It states the following:

**Theorem 3.7.** *Let $F(\sqrt[2^{n+1}]{c})$ for $c \in F$ and $n$ a positive integer. Then, we have*

$$H_2^{m+1}(F(\sqrt[2^{n+1}]{c})/F) = \sum_{k=1}^{n}\sum_{j=1}^{2^{k-1}} \overline{c^{2j-1}(\mathrm{d}\Omega_F^{m-1})^{[2^k]}} + \overline{\Omega_F^{m-1}\wedge \mathrm{d}c}.$$

Now we proceed with the proof of Theorem 1.4. Recall that $K = F(\alpha)$ and $L = F(\beta)$, where $\alpha^2 + \alpha = a \in F \setminus \wp(F)$ such that $a \in F^{2^l}$ for some $l > n$, and $\beta^{2^{n+1}} = b \in F \setminus F^2$.

First we characterize the elements in the kernel $H_2^{m+1}(K \cdot L/F)$.

Let $w \in \Omega_F^m$ be such that $\overline{w} \in H_2^{m+1}(K \cdot L/F)$. Then, $\overline{w}_K \in H_2^{m+1}(K \cdot L/K)$.

By Theorem 3.7, above, there are $\lambda_0, \lambda_{k,j} \in \Omega_K^{m-1}$ and $u \in \Omega_K^m, v \in \Omega_K^{m-1}$ such that

$$w_K = \sum_{k=1}^{n}\sum_{j=1}^{2^{k-1}} b^{2j-1}(\mathrm{d}\lambda_{k,j})^{[2^k]} + \lambda_0 \wedge \mathrm{d}b + \wp(u) + \mathrm{d}v. \tag{6}$$

Since $K/F$ is a separable extension, we can write $\lambda_0 = \epsilon_0 + \alpha\theta_0$, $\lambda_{k,j} = \epsilon_{k,j} + \alpha\theta'_{k,j}$ with $\epsilon_0, \theta_0, \epsilon_{k,j}, \theta'_{k,j} \in \Omega_F^{m-1}$, and also $u = u_1 + \alpha u_2, v = v_1 + \alpha v_2$ with $u_1, u_2 \in \Omega_F^m$, and $v_1, v_2 \in \Omega_F^{m-1}$.

We have $\mathrm{d}\lambda_{k,j} = \mathrm{d}\epsilon_{k,j} + \alpha \mathrm{d}\theta'_{k,j}$ because $\alpha$ is a square, and so

$$(\mathrm{d}\lambda_{k,j})^{[2^k]} = (\mathrm{d}\epsilon_{k,j})^{[2^k]} + \alpha^{2^k}(\mathrm{d}\theta'_{k,j})^{[2^k]}. \tag{7}$$

Now using that $\alpha^{2^k} = \alpha + \sum_{t=0}^{k-1} a^{2^t}$, we get

$$\alpha^{2^k}(\mathrm{d}\theta'_{k,j})^{[2^k]} = \alpha(\mathrm{d}\theta'_{k,j})^{[2^k]} + \sum_{t=0}^{k-1} a^{2^t}(\mathrm{d}\theta'_{k,j})^{[2^k]}.$$

Since $a \in F^{2^l}$ with $l > n$, we can write $a^{2^t}(\mathrm{d}\theta'_{k,j})^{[2^k]} = (\mathrm{d}\theta''_{k,j})^{[2^k]}$ for some $\mathrm{d}\theta''_{k,j} \in \Omega_F^{m-1}$. Therefore equation (7) can be written as

$$(\mathrm{d}\lambda_{k,j})^{[2^k]} = (\mathrm{d}\delta_{k,j})^{[2^k]} + \alpha(\mathrm{d}\theta_{k,j})^{[2^k]},$$

where $\delta_{k,j}, \theta_{k,j} \in \Omega_F^{m-1}$. Now we rewrite equation (6) as

$$\begin{aligned}
w_K &= \sum_{k=1}^{n}\sum_{j=1}^{2^{k-1}} b^{2j-1}\left((\mathrm{d}\delta_{k,j})^{[2^k]} + \alpha(\mathrm{d}\theta_{k,j})^{[2^k]}\right) + \lambda_0 \wedge \mathrm{d}b + \wp(u) + \mathrm{d}v \\
&= \sum_{k=1}^{n}\sum_{j=1}^{2^{k-1}} b^{2j-1}(\mathrm{d}\delta_{k,j})^{[2^k]} + \alpha\sum_{k=1}^{n}\sum_{j=1}^{2^{k-1}} b^{2j-1}(\mathrm{d}\theta_{k,j})^{[2^k]} + \epsilon_0 \wedge \mathrm{d}b + \\
&\quad + \alpha\theta_0 \wedge \mathrm{d}b + \wp(u_1) + au_2^{[2]} + \alpha\wp(u_2) + \mathrm{d}v_1 + \alpha \mathrm{d}v_2 \\
&= \sum_{k=1}^{n}\sum_{j=1}^{2^{k-1}} b^{2j-1}(\mathrm{d}\delta_{k,j})^{[2^k]} + \epsilon_0 \wedge \mathrm{d}b + \wp(u_1) + au_2^{[2]} + \mathrm{d}v_1 + \\
&\quad +\alpha\left(\sum_{k=1}^{n}\sum_{j=1}^{2^{k-1}} b^{2j-1}(\mathrm{d}\theta_{k,j})^{[2^k]} + \theta_0 \wedge \mathrm{d}b + \wp(u_2) + \mathrm{d}v_2\right).
\end{aligned}$$

Since $w$ is defined over $F$ we have two equations:

$$w = \sum_{k=1}^{n}\sum_{j=1}^{2^{k-1}} b^{2j-1}(\mathrm{d}\delta_{k,j})^{[2^k]} + \epsilon_0 \wedge \mathrm{d}b + \wp(u_1) + au_2^{[2]} + \mathrm{d}v_1 \tag{8}$$

$$0 = \sum_{k=1}^{n}\sum_{j=1}^{2^{k-1}} b^{2j-1}(\mathrm{d}\theta_{k,j})^{[2^k]} + \theta_0 \wedge \mathrm{d}b + \wp(u_2) + \mathrm{d}v_2. \tag{9}$$

Now we consider each term $b^{2j-1}(\mathrm{d}\theta_{k,j})^{[2^k]}$ in equation (9) over the field $L = F(\beta)$. For $k, j$ satisfying $1 \le k \le n$ and $1 \le j \le 2^{k-1}$, let $x_{l,j} =$

$\beta^{2(2j-1)2^{n-l}}(\mathrm{d}\theta_{k,j})^{[2^{k-l}]}$ for each $1 \leq l \leq k$. By applying successively the operator $\wp$, we get

$$\begin{aligned}
b^{2j-1}(\mathrm{d}\theta_{k,j})^{[2^k]} &= (\beta^{2^{n+1}})^{2j-1}(\mathrm{d}\theta_{k,j})^{[2^k]} \\
&= \beta^{2(2j-1)2^n}(\mathrm{d}\theta_{k,j})^{[2^k]} \\
&\equiv \wp(x_{1,j}) + x_{1,j} \pmod{\mathrm{d}\Omega^{m-1}_{F(\beta)} \cap (F(\beta^2)\cdot\Omega^m_F)_{F(\beta)}} \\
&\equiv \wp(x_{1,j}) + \wp(x_{2,j}) + x_{2,j} \pmod{\mathrm{d}\Omega^{m-1}_{F(\beta)} \cap (F(\beta^2)\cdot\Omega^m_F)_{F(\beta)}} \\
&\equiv \vdots \\
&\equiv \wp(\sum_{l=1}^{k} x_{l,j}) + \beta^{2(2j-1)2^{n-k}}\mathrm{d}\theta_{k,j} \pmod{\mathrm{d}\Omega^{m-1}_{F(\beta)} \cap (F(\beta^2)\cdot\Omega^m_F)_{F(\beta)}}.
\end{aligned}$$

Since $\beta^{2(2j-1)2^{n-k}}\mathrm{d}\theta_{k,j} = \mathrm{d}(\beta^{2(2j-1)2^{n-k}}\theta_{k,j}) \in \mathrm{d}\Omega^{m-1}_{F(\beta)} \cap (F(\beta^2)\cdot\Omega^m_F)_{F(\beta)}$, it follows that

$$\sum_{k=1}^{n}\sum_{j=1}^{2^{k-1}} b^{2j-1}(\mathrm{d}\theta_{k,j})^{[2^k]} = \wp(\lambda) + \mathrm{d}\tau, \tag{10}$$

where

$$\lambda = \sum_{k=1}^{n}\sum_{j=1}^{2^{k-1}}\sum_{l=1}^{k} x_{l,j} \in F(\beta^2)\cdot\Omega^m_F \quad \text{and} \quad \mathrm{d}\tau \in \mathrm{d}\Omega^{m-1}_{F(\beta)} \cap (F(\beta^2)\cdot\Omega^m_F)_{F(\beta)}. \tag{11}$$

Since $\mathrm{d}b = 0$ over $F(\beta)$, and inserting (10) into equation (9), we get

$$0 = \wp(u_2 + \lambda) + \mathrm{d}(v_2 + \tau). \tag{12}$$

This last equation implies that there is $\eta_w \in \nu_{F(\beta)}(m)$ such that

$$u_2 + \lambda = \eta_w. \tag{13}$$

Since $u_2 \in \Omega^m_F$ and $\lambda \in (F(\beta^2)\cdot\Omega^m_F)_{F(\beta)}$, we see that $\eta_w \in \nu_{F(\beta)}(m) \cap (F(\beta^2)\cdot\Omega^m_F)_{F(\beta)}$. Hence, by Proposition 3.3, $\eta_w = \eta' + \sum_{j=0}^{n-2}(\mathrm{d}s_j)^{[2^j]}$, where $\eta' \in G^{n-1}_L(m)$ and $\mathrm{d}s_0, \ldots, \mathrm{d}s_{n-2} \in \mathrm{d}\Omega^{m-1}_{F(\beta)} \cap (F(\beta^2)\cdot\Omega^m_F)_{F(\beta)}$.

Thus, equation (13) becomes $u_2 = \eta' + \lambda + \sum_{j=0}^{n-2} (\mathrm{d}s_j)^{[2^j]}$. Inserting $u_2$ into equation (8) and working over $\Omega^m_{F(\beta)}$, we get

$$
\begin{aligned}
w &= \sum_{k=1}^{n} \sum_{j=1}^{2^{k-1}} b^{2j-1} (\mathrm{d}\delta_{k,j})^{[2^k]} + \epsilon_0 \wedge \mathrm{d}b + \wp(u_1) + \\
&\quad + a(\eta' + \lambda + \sum_{j=0}^{n-2} (\mathrm{d}s_j)^{[2^j]})^{[2]} + \mathrm{d}v_1 \\
&= \sum_{k=1}^{n} \sum_{j=1}^{2^{k-1}} b^{2j-1} (\mathrm{d}\delta_{k,j})^{[2^k]} + \epsilon_0 \wedge \mathrm{d}b + \wp(u_1) + \\
&\quad + a\eta'' + a\lambda^{[2]} + a \sum_{j=0}^{n-2} (\mathrm{d}s_j)^{[2^{j+1}]} + a\mathrm{d}t + \mathrm{d}v_1,
\end{aligned} \tag{14}
$$

for some $\eta'' \in G^n_L(m)$ and $\mathrm{d}t$ coming from the squaring of $\eta'$. It is clear that $\mathrm{d}t$ belongs to $\mathrm{d}\Omega^{m-1}_{F(\beta)} \cap (F(\beta^2) \cdot \Omega^m_F)_{F(\beta)}$.

Now we look for the $F$-components of equation (14). Clearly, the elements $w$, $a\eta''$, $\wp(u_1)$, $\epsilon_0 \wedge \mathrm{d}b$ and $\mathrm{d}v_1$ are defined over $F$, and according to Lemmas 3.5 and 3.6, the $F$-component of $a \sum_{j=0}^{n-2} (\mathrm{d}s_j)^{[2^{j+1}]} + a\mathrm{d}t$ is contained in

$$
a \sum_{k=1}^{n} \sum_{j=1}^{2^{k-1}} b^{2j-1} (\mathrm{d}\Omega^{m-1}_F)^{[2^k]}. \tag{15}
$$

Moreover, by equation (11), we get

$$
\lambda^{[2]} = (\sum_{k=1}^{n} \sum_{j=1}^{2^{k-1}} \sum_{l=1}^{k} \beta^{(2j-1)2^{n+2-l}} (\mathrm{d}\theta_{k,j})^{[2^{k-l+1}]}) + \mathrm{d}t' \tag{16}
$$

for a suitable $\mathrm{d}t' \in \mathrm{d}\Omega^{m-1}_{F(\beta)} \cap (F(\beta^2) \cdot \Omega^m_F)_{F(\beta)}$. The $F$-component of the triple sum in (16) is obtained for $l = 1$, and thus the $F$-component of $a\lambda^{[2]}$ is equal to

$$
a \sum_{k=1}^{n} \sum_{j=1}^{2^{k-1}} b^{2j-1} (\mathrm{d}\theta_{k,j})^{[2^k]} + a\mathrm{d}t_0, \tag{17}
$$

where $\mathrm{d}t_0 \in \mathrm{d}\Omega^{m-1}_F$. Since the scalar $a$ belongs to $F^{2^l}$ with $l > n$, we may assume it is inside $(\mathrm{d}\Omega^{m-1}_F)^{[2^k]}$ in the double sums (15) and (17). Summarizing we see that

$$
w \in \sum_{k=1}^{n} \sum_{j=1}^{2^{k-1}} b^{2j-1} (\mathrm{d}\Omega^{m-1}_F)^{[2^k]} + \Omega^{m-1}_F \wedge \mathrm{d}b + aG^n_L(m) + \wp(\Omega^m_F) + \mathrm{d}\Omega^{m-1}_F.
$$

Summing up, we have shown

$$H_2^{m+1}(K\cdot L/F) \;\subseteq\; \sum_{k=1}^{n}\sum_{j=1}^{2^{k-1}} \overline{b^{2j-1}(\mathrm{d}\Omega_F^{m-1})^{[2^k]}} + \overline{\Omega_F^{m-1}\wedge \mathrm{d}b} + \overline{aG_L^n(m)}.$$

For the reverse inclusion, we know that $\sum_{k=1}^{n}\sum_{j=1}^{2^{k-1}} \overline{b^{2j-1}(\mathrm{d}\Omega_F^{m-1})^{[2^k]}} + \overline{\Omega_F^{m-1}\wedge \mathrm{d}b} \subset H_2^{m+1}(K\cdot L/F)$, and by Lemma 3.1 $\overline{aG_L^n(m)} \subset H_2^{m+1}(K\cdot L/F)$ because $a \in F^{2^n}$.

## 4. A DESCENT RESULT

Let $L$ be a finite purely inseparable extension of $F$. The extension $L/F$ is called *modular* if $L = F(\beta_1,\cdots,\beta_s) \cong F(\beta_1)\otimes_F\cdots\otimes_F F(\beta_s)$ for some $s\geq 1$, where for each $1\leq i\leq s$ there exist $b_i\in F\setminus F^2$ and non-negative integer $n_i$ such that $\beta_i^{2^{n_i}} = b_i$. In this case, we have $[L:F] = 2^{n_1+\cdots+n_s}$, which is also equivalent to the fact that $b_1,\cdots,b_s$ are 2-independent, i.e., $\{b_1,\cdots,b_s\}$ is a part of a 2-basis of $F$. Of course not all finite purely inseparable extensions are modular, see for instance [16, Example 1.1, page 405]. We refer to [16] for other equivalent definitions of modularity.

The aim of this section is to give a general result (Proposition 4.4) that is needed in the proofs of Theorems 1.2 and 1.5. This result applies not only to simple purely inseparable extensions but also to modular purely inseparable extensions.

**Notation 4.1.** *Suppose that as above $L = F(\beta_1,\ldots,\beta_s)$ is a modular purely inseparable extension. We denote by $L^{(2)}$ the subfield of $L$ given by $L^{(2)} = F(\beta_1^2,\ldots,\beta_s^2)$.*

Let $K = F(\alpha_1,\alpha_2)$ be a separable biquadratic extension such that $\alpha_i^2+\alpha_i = a_i \in F\setminus\wp(F)$ for $1\leq i\leq 2$. Without loss of generality, we may suppose that $a_1,a_2\in F^2$. Set $M = L^{(2)}(\alpha_2)$. Since $L/F$ is modular, the set $\{b_1,\ldots,b_s\}$ can be completed to a 2-basis $\mathcal{B}\cup\{b_1,b_2,\ldots,b_s\}$ of $F$. Then, $\mathcal{C} := \mathcal{B}\cup\{\beta_1,\ldots,\beta_s\}$ is a 2-basis of $L$. Put $\mathcal{C} = \{e_i \mid i\in I\}$ and fix an ordering on $\mathcal{C}$ so that $\beta_1<\cdots<\beta_s$ and $e_i<\beta_j$ for all $e_i\in\mathcal{B}$ and $1\leq j\leq s$.

We will need the following Lemmas.

**Lemma 4.2.** *Keep the same notations and hypotheses as before. Let $L' = L(\alpha_2)$, $\gamma\in\Sigma_{m,L'}$ and $u\in(\Omega_M^m)_{L'}$ nonzero such that $\gamma = max(u)$. Then, there exist $x\in M$, $e_{\gamma(1)},\cdots,e_{\gamma(m)}\in\mathcal{B}$ and $u'\in(\Omega_M^m)_{L',<\gamma}$ such that $u_{L'} = x\frac{\mathrm{d}e_\gamma}{e_\gamma}+u'$.*

*Proof.* Since $u\in(\Omega_M^m)_{L'}$, it is clear that $u\in(M\cdot\Omega_F^m)_{L'}$. Moreover, $\mathcal{B}\cup\{b_1,b_2,\ldots,b_s\}$ is a 2-basis of $F$ and $b_1,\ldots,b_s$ are squares in $L$, it follows that the expression of $u$ contains only terms of the form $(c\frac{\mathrm{d}e_\gamma}{e_\gamma})_{L'}$, where $e_{\gamma(1)},\ldots,e_{\gamma(m)}\in\mathcal{B}$ and $c\in M$. Hence the lemma. □

Izhboldin in [12] introduced the groups $Q^m(F,n)$ for $n > 1$ whose definition is based on $p$-Witt vectors of length $n$. Here, we will only consider the groups $Q^m(F,1)$ defined as follows. For any integer $m \geq 1$, let $Q^m(F,1)$ be the group $F \otimes F^{*\otimes m}/J_{m,1}$, where $J_{m,1}$ is the subgroup generated by the elements $a \otimes x_1 \otimes \cdots \otimes x_m$ such that $x_i = x_j$ for some $i \neq j$, and the elements $a^s \otimes a \otimes x_2 \otimes \cdots \otimes x_n$ such that $a \in F^*$ and $s$ an integer $\geq 1$. We have a well-defined Artin-Schreier homomorphism $\wp : Q^m(F,1) \longrightarrow Q^m(F,1)$ defined on generators by:

$$\wp(\overline{a \otimes x_1 \otimes \cdots \otimes x_m}) = \overline{(a^2 - a) \otimes x_1 \otimes \cdots \otimes x_m}, \tag{18}$$

and whose kernel is isomorphic to $\nu_F(m)$ see [12]. Furthermore, Izhboldin proved that $Q^m(F,1) \simeq \Omega_F^m/B_\infty^m$, where $B_\infty^m = \cup_{r\geq 1} B_r^m F$, and $B_r^m F$ is the subgroup of $\Omega_F^m$ generated by the elements $(f_1 f_1 \cdots f_m)^{2^k} \frac{\mathrm{d}f_1}{f_1} \wedge \cdots \wedge \frac{\mathrm{d}f_m}{f_m}$ with $k < r$. Note that $B_0^m F = 0$ and $B_1^m F = \mathrm{d}\Omega_F^{m-1}$.

In the computations that follow, especially those in the proof of Lemma 4.3, we will work with the quotient groups $\Omega_F^m/B_\infty^m$ instead of the groups $Q^m(F,1)$.

For $K$ the separable biquadratic extension as above, let us set $K_1 = F(\alpha_1)$, $K_2 = F(\alpha_2)$ and $K_3 = F(\alpha_3)$ where $\alpha_3 = \alpha_1 + \alpha_2$. We consider the trace maps $T_i = \mathrm{Tr}_{K_i/F}$, $\tilde{T}_i = \mathrm{Tr}_{K/K_i}$ and $i_j : Q^m(F,1) \hookrightarrow Q^m(K_j,1)$. Aravire and Jacob showed in [6, Theorem 8] that the following sequence is exact:

$$Q^m(F,1) \xrightarrow{d_1} Q^m(K,1) \oplus Q^m(F,1) \xrightarrow{d_2} \oplus_{i=1}^3 Q^m(K_i,1) \xrightarrow{d_3} Q^m(F,1)^{\oplus 2} \tag{19}$$

where the maps $d_1, d_2$ and $d_3$ are defined as follows:

$$\begin{aligned}
d_1 &= i_{K/F} \oplus -2 \cdot 1_F, \\
d_2 &= (\tilde{T}_1 + i_1, \tilde{T}_2 + i_2, \tilde{T}_3 + i_3), \\
d_3 &= (T_1 - T_3, T_2 - T_3).
\end{aligned}$$

Additionally the same authors showed in [5, Theorem 34 and Remark 5] that the following sequence for the groups $\nu_F(m)$ is exact:

$$\nu_K(m) \oplus \nu_F(m) \xrightarrow{d_2^*} \oplus_{i=1}^3 \nu_{K_i}(m) \xrightarrow{d_3^*} \nu_F(m)^{\oplus 2} \tag{20}$$

where the maps $d_1^*, d_2^*$ and $d_3^*$ are defined in the same way as $d_1, d_2$ and $d_3$.

We have the following.

**Lemma 4.3.** *Let $K/F$ be the separable biquadratic extension as before, $\gamma \in \Sigma_{m,F}$, $w \in \Omega_F^m$, $u = u_0 + \alpha_1 u_1 + \alpha_2 u_2 + \alpha_1\alpha_2 u_3$ with $u_i \in \Omega_F^m$ and $v \in \Omega_K^{m-1}$. Suppose that $max(u_1) < \gamma$, $max(u_3) < \gamma$ and*

$$w_K = \wp(u) + \mathrm{d}(v) \in \Omega_K^m. \tag{21}$$

*Then, there exists $\delta \in \nu_K(m)$, $u_0', u_1', u_2' \in \Omega_F^m$ such that $u = \delta + u_0' + \alpha_1 u_1' + \alpha_2 u_2'$ and $max(u_1') < \gamma$.*

*Proof.* Along this proof we will write $[\eta]$ for the class of $\eta \in \Omega_F^m$ in the quotient $\Omega_F^m/B_\infty^m$. Accordingly, since $\nu_F(m)$ is isomorphic to the kernel of the Artin-Schreier homomorphism given in (18), we may identify the elements of $\nu_F(m)$ with their corresponding classes $[\eta]$ in $\Omega_F^m/B_\infty^m$.

Applying the trace maps $\tilde{T_1}$, $\tilde{T_2}$ and $\tilde{T_3}$ to equation (21) and using [1, Proposition 2.8], we get

$$\begin{aligned}
\wp(\tilde{T_1}(u)) &= \wp(u_2+\alpha_1 u_3) \in \mathrm{d}\Omega^{m-1}_{F(\alpha_1)} \\
\wp(\tilde{T_2}(u)) &= \wp(u_1+\alpha_2 u_3) \in \mathrm{d}\Omega^{m-1}_{F(\alpha_2)} \\
\wp(\tilde{T_3}(u)) &= \wp(u_1+u_2+u_3+\alpha_3 u_3) \in \mathrm{d}\Omega^{m-1}_{F(\alpha_3)}.
\end{aligned}$$

Hence, $\eta_1 := u_2+\alpha_1 u_3 \in \nu_{F(\alpha_1)}(m)$, $\eta_2 := u_1+\alpha_2 u_3 \in \nu_{F(\alpha_2)}(m)$ and $\eta_3 := u_1+u_2+u_3+\alpha_3 u_3 \in \nu_{F(\alpha_3)}(m)$. Clearly, $([\eta_1],[\eta_2],[\eta_3]) \in \mathrm{Ker} d_3$. On the one hand, for $\eta = \alpha_1 u_1+\alpha_2 u_2+\alpha_1\alpha_2 u_3$, the element $[\eta] = [\alpha_1 u_1+\alpha_2 u_2+\alpha_1\alpha_2 u_3] \in Q^m(K,1)$ satisfies $d_2([\eta],0) = ([\eta_1],[\eta_2],[\eta_3])$. On the other hand, by sequence (20), there exists $([\delta],[\epsilon]) \in \nu_K(m)\oplus\nu_F(m)$ such that $d_2^*([\delta],[\epsilon]) = ([\eta_1],[\eta_2],[\eta_3])$. Hence, $([\eta+\delta],[\epsilon]) \in \mathrm{Ker} d_2$. Consequently, by sequence (19), there exists $[\epsilon'] \in Q^m(F,1)$ such that $([\eta+\delta],[\epsilon]) = ([\epsilon'],0)$. Since $[\delta] \in \nu_K(m)$, it follows that $\wp(\eta)+\wp(\epsilon') \in B_\infty^m$ and thus $\wp(\eta)+\wp(\epsilon') \in B_r^m K$ for some $r \geq 1$. One has $\wp(\eta) = \eta' + \mathrm{d}(z)$, where $\eta' = a_1 u_1^{[2]} + a_2 u_2^{[2]} + a_1 a_2 u_3 + \alpha_1(\wp(u_1)+a_2 u_3) + \alpha_2(\wp(u_2)+a_1 u_3)$ and $z \in \Omega_K^{m-1}$. Hence, $\eta'+\wp(\epsilon') \in B_r^m K$. We write $K = F(\alpha_1^{2^r}, \alpha_2^{2^r})$. It follows from [6, Lemma 1] that

$$\eta'+\wp(\epsilon') = w_0 + \alpha_1^{2^r} w_1 + \alpha_2^{2^r} w_2 + \alpha_1^{2^r}\alpha_2^{2^r} w_3, \tag{22}$$

for suitable $w_0, w_1, w_2, w_3 \in B_r^m F$.

Since $\eta'+\wp(\epsilon')$ has no factor of $\alpha_1\alpha_2$, it follows that $w_3 = 0$. Moreover, since the factor of $\alpha_1$ in the left hand side of (22) has multi-index smaller that $\gamma$, then the maximal multi-index of $w_1$ must be smaller than $\gamma$.

Notice that for the generators of the group $B_r^m F$, the following equality holds for each $s < r$:

$$\alpha_i^{2^r}(f_1 f_2\cdots f_m)^{2^s}\frac{\mathrm{d}f_1}{f_1}\wedge\cdots\wedge\frac{\mathrm{d}f_m}{f_m} = \left(\alpha_i^{2^{r-s}} f_1 f_2\cdots f_m\right)^{2^s}\frac{\mathrm{d}(\alpha_i^{2^{r-s}}f_1)}{(\alpha_i^{2^{r-s}}f_1)}\wedge\cdots\wedge\frac{\mathrm{d}f_m}{f_m}.$$

Now, applying several times the Artin-Schreier operator to the right hand side of (22), it is easy to see that there exists $S \in \Omega_K^m$ such that

$$\eta'+\wp(\epsilon') \equiv \wp(S) \pmod{\mathrm{d}\Omega_K^{m-1}}.$$

Hence, $\wp(\eta+S+\epsilon') \in \mathrm{d}\Omega_K^{m-1}$, which implies that $\delta := \eta+S+\epsilon' \in \nu_K(m)$.

Notice that the factor of $\alpha_1$ in $S$ comes from $\alpha_1^{2^r} w_1$, and thus it has maximal multi-index smaller than $\gamma$. Hence, the maximal multi-index of $\delta$ is also smaller than $\gamma$.

Now to conclude we have $\tilde{T_1}(u) = \eta_1 = \tilde{T_1}(\eta)$, and thus $\mathrm{Tr}_{K/F}(u+\eta) = 0$, which implies that $u = \eta + x_0 + \alpha_1 x_1 + \alpha_2 x_2$ for some $x_0, x_1, x_2 \in \Omega_F^m$. Consequently,

$u = \delta + S + \epsilon' + x_0 + \alpha_1 x_1 + \alpha_2 x_2$. The maximal multi-index of $x_1$ is smaller than $\gamma$ because the factors of $\alpha_1$ in the terms $u$, $\delta$ and $S + \epsilon'$ are also smaller than $\gamma$. Since $S + \epsilon'$ has no factor of $\alpha_1 \alpha_2$, we get $S + \epsilon' + x_0 + \alpha_1 x_1 + \alpha_2 x_2 = u'_0 + \alpha_1 u'_1 + \alpha_2 u'_2$ for suitable $u'_0, u'_1, u'_2 \in \Omega^m_F$ and $max(u'_1) < \gamma$. □

Now we give a crucial result for the proof of Theorem 1.5.

**Proposition 4.4.** *Let $w \in \Omega^m_{L^{(2)}}$, $\lambda_1, \lambda_2 \in \nu_L(m)$, $u \in \Omega^m_L$ and $v \in \Omega^{m-1}_L$ be such that*

$$w_L + a_1\lambda_1 + a_2\lambda_2 = \wp(u) + \mathrm{d}v \ \text{ in } \Omega^m_L. \tag{23}$$

*Then, there exist $\delta_1, \delta_2 \in \nu_{L^{(2)}}(m)$, $u_1 \in \Omega^m_{L^{(2)}}$ and $v_1 \in \Omega^{m-1}_{L^{(2)}}$ such that*

$$w_{L^{(2)}} + a_1\delta_1 + a_2\delta_2 = \wp(u_1) + \mathrm{d}v_1 \ \text{ in } \Omega^m_L.$$

*Proof.* Let $M = L^{(2)}(\alpha_2)$ as before and $L' = L(\alpha_2)$. Since $F(\alpha_2)/F$ is separable, it follows that $\mathcal{B} \cup \{b_1, \dots, b_s\}$ (*resp.* $\mathcal{C} \cup \{\beta_1, \dots, \beta_s\}$) remains a 2-basis of $F(\alpha_2)$ (*resp.* a 2-basis of $L(\alpha_2)$). For our proof we proceed in two steps.

**Step 1.** We will prove the existence of $\widetilde{\lambda_1} \in \nu_{L^{(2)}}(m)$, $\widetilde{u} \in \Omega^m_M$ and $\widetilde{v} \in \Omega^{m-1}_M$ such that

$$w_M + a_1\widetilde{\lambda_1} = \wp(\widetilde{u}) + \mathrm{d}(\widetilde{v}) \ \text{ in } \Omega^m_{L'}. \tag{24}$$

Since $(a_2\lambda_2)_{L'} \equiv \wp(\alpha_2\lambda_2) \pmod{\mathrm{d}\Omega^{m-1}_{L'}}$, equation (23) becomes

$$w_{L'} + (a_1\lambda_1)_{L'} = \wp(\theta) + \mathrm{d}(v') \ \text{ in } \Omega^m_{L'}, \tag{25}$$

where $v' \in \Omega^{m-1}_{L'}$ and $\theta = u + \alpha_2\lambda_2$.

Our aim is to descend $\lambda_1$ to the field $L^{(2)}$, and $\theta$ and $v'$ to $M$. In fact, as in equation (23), let $\gamma, \tau \in \Sigma_{m,L'}$ be such that $\gamma = max(w_{L'} + a_1(\lambda_1)_{L'})$ and $\tau = max(\theta)$. By Lemma 2.5, we may suppose that $\tau \leq \gamma$ in the three cases that we consider below.

**(1)** Suppose that $max(w_{L'}) > max((\lambda_1)_{L'})$. Then, $\gamma = max(w_{L'})$, and by Lemma 4.2 we have $w_{L'} = l\frac{de_\gamma}{e_\gamma} + w'$ such that $l \in M$, $e_{\gamma(i)} \in \mathcal{B}$ and $w' \in (\Omega^m_M)_{L',<\gamma}$.

We write $\theta = r\frac{\mathrm{d}e_\gamma}{e_\gamma} + \theta'$ and $\mathrm{d}(v') = s\frac{de_\gamma}{e_\gamma} + v''$ such that $r, s \in L'$ and $\theta', v'' \in \Omega^m_{L',<\gamma}$. Moreover, by Lemma 2.2, $s = \sum_{i=1}^m s_i e_{\gamma(i)}$ such that $s_i \in L'_{<\gamma(i)} = L'^2(e_j \mid j < \gamma(i))$ for all $1 \leq i \leq m$. Since $e_{\gamma(1)}, \dots, e_{\gamma(m)} \in \mathcal{B}$, it follows that $s_1, \dots, s_m \in M$. We deduce from Lemma 2.6 that $s\frac{\mathrm{d}e_\gamma}{e_\gamma} = \mathrm{d}(v_0) + v_1$, where $v_0 \in (\Omega^{m-1}_M)_{L'}$ and $v_1 \in \Omega^m_{L',<\gamma}$. Hence, $\mathrm{d}(v) = \mathrm{d}(v_0)_{L'} + \mathrm{d}(v_2)$, where $\mathrm{d}(v_2) = v' + \mathrm{d}(v_1) \in \mathrm{d}\Omega^{m-1}_{L',<\gamma}$.

Taking coefficients of $\frac{\mathrm{d}e_\gamma}{e_\gamma}$ from equation (23) yields $l = \wp(r) + s$. Since $l, s \in M$, it follows that $r \in M$. Equation (23) becomes $(w + \wp(r\frac{\mathrm{d}e_\gamma}{e_\gamma}) + \mathrm{d}(v_0))_{L'} + (a_1\lambda_1)_{L'} =$

$\wp(\theta') + \mathrm{d}(v_2)$, and thus the multi-index $max(w_{L'})$ reduces after changing $w_{L'}$ modulo the group $\wp((\Omega_M^m)_{L'}) + \mathrm{d}((\Omega_M^{m-1})_{L'})$.

**(2)** Suppose that $max(w_{L'}) = max((\lambda_1)_{L'})$. Since $w \in \Omega_M^m$ and $\gamma$ is given by $w_{L'}$, we get $e_{\gamma(1)}, \ldots, e_{\gamma(n)} \in \mathcal{B}$ (Lemma 4.2). As the extension $L'/L$ is separable, this implies that the maximal multi-index of $\lambda_1$ over $L$ is also $\gamma$. By Kato's decomposition over $L$, we get $\lambda_1 = \frac{df_{\gamma(1)}}{f_{\gamma(1)}} \wedge \cdots \wedge \frac{df_{\gamma(m)}}{f_{\gamma(m)}} + \lambda_1'$ such that $f_{\gamma(i)} \in L_{\gamma(i)}$ and $\lambda_1' \in \nu_{L,<\gamma}(m)$. We have $f_{\gamma(i)} \in L^{(2)}$ because $e_{\gamma(1)}, \cdots, e_{\gamma(m)} \in \mathcal{B}$, and thus $\frac{df_\gamma}{f_\gamma} := \frac{df_{\gamma(1)}}{f_{\gamma(1)}} \wedge \cdots \wedge \frac{df_{\gamma(m)}}{f_{\gamma(m)}} \in \nu_{L^{(2)}}(m)$. Now we consider the new 2-basis $\mathcal{C}' = \{e_i' \mid i \in I\}$ of $L$ defined as follows:

$$e_i' = \begin{cases} e_i & \text{if } i \neq \gamma(1), \ldots, \gamma(m) \\ f_i & \text{if } i \in \{\gamma(1), \ldots, \gamma(m)\}, \end{cases}$$

without changing the ordering of $I$ (Proposition 2.1). Note that $\mathcal{C}'$ is a 2-basis of $L'$.

We write $w = l\frac{df_\gamma}{f_\gamma} + w'$, $\theta = r\frac{df_\gamma}{f_\gamma} + \theta'$ and $\mathrm{d}(v') = s\frac{df_\gamma}{f_\gamma} + v''$ such that $l, r, s \in L'$ and $w', \theta', v'' \in \Omega^m_{L',<\gamma}$. Since $w, (\frac{df_\gamma}{f_\gamma})_M \in \Omega_M^m$, it follows that $l \in M$ and thus $w' \in (\Omega_M^m)_{L',<\gamma}$. Moreover, by Lemma 2.2, we have $s = \sum_{i=1}^m s_i f_{\gamma(i)}$ such that $s_i \in L'^2(e_j' \mid j < \gamma(i))$. Because $e_j' \in L^{(2)}$ for all $j \leq \gamma(i)$, we deduce that $s_1, \ldots, s_m \in M$, and thus $s \in M$. As in case (1.1) we write $\mathrm{d}(v') = \mathrm{d}(v_0)_{L'} + \mathrm{d}(v_1)$ such that $\mathrm{d}(v_0)_{L'} \in (\mathrm{d}\Omega_M^{m-1})_{L'}$ and $\mathrm{d}(v_1) \in \mathrm{d}\Omega^{m-1}_{L',<\gamma}$. Comparing coefficients of $\frac{\mathrm{d}f_\gamma}{f_\gamma}$ from equation (23) yields $l + a_1 = \wp(r) + s$. Note that $r \in M$ because $l, a_1, s \in M$. Now equation (23) becomes $(w + a_1\frac{\mathrm{d}f_\gamma}{f_\gamma} + \wp(r\frac{\mathrm{d}f_\gamma}{f_\gamma}) + \mathrm{d}(v_0))_{L'} + a_1\lambda_1' = \wp(\theta') + \mathrm{d}(v_1)$, and thus the multi-index $max(w_{L'})$ reduces after changing $w_{L'}$ modulo the group $a_1(\nu_{L^{(2)}}(m))_{L'} + \wp((\Omega_M^m)_{L'}) + \mathrm{d}((\Omega_M^{m-1})_{L'})$. Note that for this reduction we use the group $a_1(\nu_{L^{(2)}}(m))_{L'}$ instead of $a_1(\nu_M(m))_{L'}$, and $\lambda_1' \in \nu_{L^2}(m)$.

**(3)** Suppose that $max((\lambda_1)_{L'}) > max(w_{L'})$. Then, as in Case **(2)**, $\lambda_1$ has maximal multi-index $\gamma$ over $L$. By Kato's decomposition we have $\lambda_1 = \frac{\mathrm{d}f_\gamma}{f_\gamma} + \lambda_1'$, where $\frac{\mathrm{d}f_\gamma}{f_\gamma} = \frac{\mathrm{d}f_{\gamma(1)}}{f_{\gamma(1)}} \wedge \cdots \wedge \frac{\mathrm{d}f_{\gamma(m)}}{f_{\gamma(m)}}$, $f_{\gamma(i)} \in L_{\gamma(i)}$ and $\lambda_1' \in \nu_{L,<\gamma}(m)$. Similarly, we have $\theta = r\frac{\mathrm{d}f_\gamma}{f_\gamma} + \theta'$ for some $r \in L'$ and $\theta' \in \Omega^m_{L'}$. It follows from equation (25) that over the field $L'(\alpha_1)$ we have

$$\wp((r + \alpha_1)\frac{\mathrm{d}f_\gamma}{f_\gamma}) \in \mathrm{d}\Omega^{m-1}_{L'(\alpha_1)} + \Omega^m_{L'(\alpha_1),<\gamma}. \tag{26}$$

Lemma 2.3 implies that

$$(r + \alpha_1)\frac{\mathrm{d}f_\gamma}{f_\gamma} = \frac{\mathrm{d}g_\gamma}{g_\gamma} + \eta, \tag{27}$$

where $g_{\gamma(i)} \in L'^2(\alpha_1^2)(f_j \mid j \leq \gamma(i))$, for any $1 \leq i \leq m$, and $\eta \in \Omega^m_{L'(\alpha_1),<\gamma}$.

Applying $\wp$ to equation (27) yields

$$a_1\frac{\mathrm{d}f_\gamma}{f_\gamma} = \wp\left(r\frac{\mathrm{d}f_\gamma}{f_\gamma} + \eta\right) + \mathrm{d}z \ \text{ in } \ \Omega^{m-1}_{L'(\alpha_1)}, \tag{28}$$

where $z \in \mathrm{d}\Omega^{m-1}_{L'(\alpha_1)}$.

Since the multi-indices of the factors of $\alpha_1$ and $\alpha_1\alpha_2$ in $r\frac{\mathrm{d}f_\gamma}{f_\gamma} + \eta$ are smaller than $\gamma$ (because these factors come from $\eta \in \Omega^m_{L'(\alpha_1),<\gamma}$), we apply Lemma 4.3 to the biquadratic separable extension $L'(\alpha_1)$ to get $\delta \in \nu_{L'(\alpha_1)}(m)$ and $v_0, v_1, v_2 \in \Omega^m_L$ such that $r\frac{\mathrm{d}f_\gamma}{f_\gamma} + \eta = \delta + v_0 + \alpha_1 v_1 + \alpha_2 v_2$ and $v_1 \in \Omega^m_{L,<\gamma}$. Hence, it follows from equation (28) that

$$a_1\frac{\mathrm{d}f_\gamma}{f_\gamma} + \wp(v_0 + \alpha_1 v_1 + \alpha_2 v_2) \in \mathrm{d}\Omega^{m-1}_{L'(\alpha_1)}.$$

Consequently, equation (25) becomes

$$w_M + a_1\lambda_1' = \wp(\theta + v_0 + \alpha_1 v_1 + \alpha_2 v_2) + \mathrm{d}(z') \ \text{ in } \Omega^m_{L'(\alpha_1)} \tag{29}$$

where $z' \in \Omega^{m-1}_{L'(\alpha_1)}$.

Now using the decomposition $\Omega^m_{L'(\alpha_1)} = \Omega^m_{L'} + \alpha_1\Omega^m_{L'}$, and the fact that $w$, $a_1\lambda_1'$ and $\theta$ are defined over $L'$, we deduce the following equations in $\Omega^m_{L'}$:

$$\begin{aligned} w_M + a_1\lambda_1' &= a_1 v_1^{[2]} + \wp(\theta + v_0 + \alpha_2 v_2) + \mathrm{d}(t) \quad (30)\\ \wp(v_1) &= \mathrm{d}(t') \quad (31)\end{aligned}$$

for suitable $t, t' \in \Omega^{m-1}_{L'}$.

From equation (31) we get $v_1 \in \nu_L(m)$, and inserting it in equation (30) we obtain

$$w_M + a_1(\lambda_1' + v_1) = \wp(\theta + v_0 + \alpha_2 v_2) + \mathrm{d}(t'') \ \text{ in } \Omega^m_{L'} \ \text{ for some } t'' \in \Omega^{m-1}_{L'}.$$

Thus the multi-index reduces because $max(\lambda_1' + v_1) < \gamma$.

**Step 2.** Suppose we have $\widetilde{\lambda_1} \in \nu_{L^{(2)}}(m)$, $\widetilde{u} \in \Omega^m_M$ and $\widetilde{v} \in \Omega^{m-1}_M$ such that

$$w_M + a_1\widetilde{\lambda_1} = \wp(\widetilde{u}) + \mathrm{d}(\widetilde{v}) \in \Omega^m_{L'}. \tag{32}$$

We will derive from equation (32) elements $\delta_1, \delta_2 \in \nu_{L^{(2)}}(m), u_1 \in \Omega^m_{L^{(2)}}$ and $v_1 \in \Omega^{m-1}_{L^{(2)}}$ satisfying: $w_{L^{(2)}} + a_1\delta_1 + a_2\delta_2 = \wp(u_1) + \mathrm{d}v_1 \in \Omega^m_L$.

Recall that $\Omega^m_M = \Omega^m_{L^{(2)}} + \alpha_2\Omega^m_{L^{(2)}}$ and $\mathrm{d}\Omega^{m-1}_M = \mathrm{d}\Omega^{m-1}_{L^{(2)}} + \alpha_2\mathrm{d}\Omega^{m-1}_{L^{(2)}}$. Hence, we put

$$\begin{aligned} \widetilde{u} &= u_1 + \alpha_2 u_2\\ \mathrm{d}(\widetilde{v}) &= \mathrm{d}v_1 + \alpha_2\mathrm{d}v_2, \quad (33)\end{aligned}$$

where $u_1, u_2 \in \Omega^m_{L^{(2)}}$ and $\mathrm{d}v_1, \mathrm{d}v_2 \in \mathrm{d}\Omega^{m-1}_{L^{(2)}}$.

We have $\wp(\widetilde{u}) = \wp(u_1) + \alpha_2\wp(u_2) + a_2 u_2^{[2]}$. Inserting this expression and the second line of equation (33) into equation (32) gives:

$$w_{L'} + a_1\widetilde{\lambda_1} = \wp(u_1) + \alpha_2\wp(u_2) + a_2 u_2^{[2]} + \mathrm{d}v_1 + \alpha_2\mathrm{d}v_2.$$

Consequently,

$$w_L + a_1\widetilde{\lambda_1} = \wp(u_1) + a_2 u_2^{[2]} + \mathrm{d}v_1 \quad (34)$$

$$0 = \wp(u_2) + \mathrm{d}v_2. \quad (35)$$

Equation (35) implies that $u_2 \in \nu_L(m) \cap (L^{(2)} \cdot \Omega_F^m)_L$. Lemma 3.2 implies that $u_2 \in (\nu_{L^{(2)}}(m))_L$. Thus, $u_2^{[2]} = u_2 + \mathrm{d}t$ for some $\mathrm{d}t \in (L^{(2)} \cdot \Omega_F^m)_L \cap \mathrm{d}\Omega_L^{m-1}$. Hence, $\mathrm{d}t \in \mathrm{d}\Omega_{L^{(2)}}^{m-1}$ by Corollary 3.4. Replacing this expression for $u_2^{[2]}$ into equation (34) yields

$$w_L + a_1\delta_1 + a_2\delta_2 = \wp(u_1) + \mathrm{d}z,$$

for suitable $\mathrm{d}z \in \mathrm{d}\Omega_{L^{(2)}}^{m-1}$ and $\delta_1, \delta_2 \in \nu_{L^{(2)}}(m)$. This proves the proposition. □

**Corollary 4.5.** *Let $L$ be a purely inseparable multiquadratic extension of $F$. Let $w \in \Omega_F^m$, $\lambda_1, \lambda_2 \in \nu_L(m)$, $u \in \Omega_L^m$ and $v \in \Omega_L^{m-1}$ be such that*

$$w_L + a_1\lambda_1 + a_2\lambda_2 = \wp(u) + \mathrm{d}v \in \Omega_L^m.$$

*Then, there exist $\delta_1, \delta_2 \in \nu_F(m)$, $u_1 \in \Omega_F^m$ and $v_1 \in \Omega_F^{m-1}$ such that*

$$w + a_1\delta_1 + a_2\delta_2 = \wp(u_1) + \mathrm{d}v_1 \in \Omega_L^m.$$

*Proof.* This is a direct consequence of Proposition 4.4 because $L^{(2)} = F$ when $L$ is a multiquadratic purely inseparable extension of $F$. □

## 5. Proof of Theorems 1.2 and 1.5

5.1. **Proof of Theorem 1.2.** Let $L/F$ be a finite purely inseparable multiquadratic extension and $K/F$ a biquadratic separable extension. Obviously $L^{(2)} = F$. Let us write $K = F(\alpha_1, \alpha_2)$ such that $\alpha_i^2 + \alpha_i = a_i \in F$ for $1 \leq i \leq 2$.

Clearly $H_2^{m+1}(L/F) + H_2^{m+1}(K/F) \subset H_2^{m+1}(K \cdot L/F)$. Conversely, let $w \in \Omega_F^m$ be such that $\overline{w} \in H_2^{m+1}(K \cdot L/F)$, then, $\overline{w}_L \in H_2^{m+1}(K \cdot L/L)$ and thus by [6], there exist $\lambda_1, \lambda_2 \in \nu_L(m)$, $u \in \Omega_L^m$ and $v \in \Omega_L^{m-1}$ such that

$$w_L + a_1\lambda_1 + a_2\lambda_2 = \wp(u) + \mathrm{d}(v) \in \Omega_L^m.$$

By Corollary 4.5 we may suppose that $\lambda_1, \lambda_2 \in \nu_F(m)$, $u \in \Omega_F^m$ and $v \in \Omega_F^{m-1}$. Hence, $\overline{w} + \overline{a_1\lambda_1} + \overline{a_2\lambda_2} \in H_2^{m+1}(L/F)$, i.e., $\overline{w} \in H_2^{m+1}(K/F) + H_2^{m+1}(L/F)$. This proves the theorem. □

5.2. **Proof of Theorem 1.5.** Let $K = F(\alpha_1, \alpha_2)$ be a biquadratic separable extension of $F$ such that $\alpha_i^2 + \alpha_i = a_i \in F$ for $1 \leq i \leq 2$, and $L = F(\beta)$ a simple purely inseparable extension of $F$ such that $\beta^{2^{n+1}} = b \in F \setminus F^2$. We assume that $a_1, a_2 \in F^{2^l}$ for some $l > n$.

Let $w \in \Omega_F^m$ be such that $\overline{w} \in H_2^{m+1}(K \cdot L/F)$. Then, $\overline{w}_L \in H_2^{m+1}(K \cdot L/L)$. It follows from [6] that there exist $\lambda_1, \lambda_2 \in \nu_L(m)$, $u \in \Omega_L^m$ and $v \in \Omega_L^{m-1}$ such that

$$w_L + a_1\lambda_1 + a_2\lambda_2 = \wp(u) + \mathrm{d}(v) \in \Omega_L^m. \tag{36}$$

By Proposition 4.4, we may suppose that $\lambda_1, \lambda_2 \in \nu_{F(\beta^2)}(m)$, $u \in \Omega^m_{F(\beta^2)}$ and $v \in \Omega^{m-1}_{F(\beta^2)}$. Hence, $\lambda_1, \lambda_2 \in \nu_{F(\beta)}(m) \cap (F(\beta^2) \cdot \Omega_F^m)_{F(\beta)}$, $u \in F(\beta^2) \cdot \Omega_F^m$ and $v \in F(\beta^2) \cdot \Omega_F^{m-1}$.

By Proposition 3.3, we have $\lambda_i = \widetilde{\lambda_i} + \sum_{j=0}^{n-1} (\mathrm{d}\theta_{i,j})^{[2^j]}$ where $\widetilde{\lambda_i} \in G_L^n(m)$ and $\theta_{i,j} \in \Omega_L^{m-1}$, for $1 \leq i \leq 2$.

Writing $u = \sum_{i=0}^{2^n-1} \beta^{2i}\eta_i$ where $\eta_i \in \Omega_F^m$, $0 \leq i \leq 2^n - 1$ we see that

$$\wp(u) = \sum_{i=0}^{2^n-1} (\beta^{2i}\eta_i + \beta^{4i}\eta_i^{[2]}) + \mathrm{d}t$$

for a suitable $\mathrm{d}t \in \mathrm{d}\Omega^{m-1}_{F(\beta)} \cap (F(\beta^2) \cdot \Omega_F^m)_{F(\beta)}$. Using the description of differentials in $\mathrm{d}\Omega^{m-1}_{F(\beta)} \cap (F(\beta^2) \cdot \Omega_F^m)_{F(\beta)}$ given by equation (4), we write

$$\mathrm{d}(v) + \mathrm{d}t = \sum_{i=0}^{2^n-1} \beta^{2i}\mathrm{d}v_i \ \text{ with } v_i \in \Omega_F^{m-1}, 0 \leq i \leq 2^n - 1.$$

Inserting the expressions of $\lambda_i$, $1 \leq i \leq 2$, $\mathrm{d}(v) + \mathrm{d}t$ and $\wp(u)$ in equation (36), we get in $\Omega_L^m$ the following:

$$\begin{aligned} w_L + a_1\widetilde{\lambda_1} + a_2\widetilde{\lambda_2} &= a_1\sum_{j=0}^{n-1}(\mathrm{d}\theta_{1,j})^{[2^j]} + a_2\sum_{j=0}^{n-1}(\mathrm{d}\theta_{2,j})^{[2^j]} \\ &+ \sum_{i=0}^{2^n-1}(\beta^{2i}\eta_i + \beta^{4i}\eta_i^{[2]}) + \sum_{i=0}^{2^n-1}\beta^{2i}\mathrm{d}v_i. \end{aligned} \tag{37}$$

Furthermore, by equation (5), we have

$$\begin{aligned} a_1\sum_{j=0}^{n-1}(\mathrm{d}\theta_{1,j})^{[2^j]} + a_2\sum_{j=0}^{n-1}(\mathrm{d}\theta_{2,j})^{[2^j]} &= \sum_{j=0}^{n-1}(\mathrm{d}\xi_{j0})^{[2^j]} + \\ &+ \sum_{j=0}^{n-1}\sum_{i=1}^{2^n-1}\beta^{i2^{j+1}}(\mathrm{d}\xi_{j,2i})^{[2^j]}, \end{aligned} \tag{38}$$

where $\mathrm{d}\xi_{j,i} \in \mathrm{d}\Omega_F^{m-1}$ for $0 \leq j \leq n-1$ and $0 \leq i \leq 2^n - 1$ (of course we may put $a_i$ inside $(\mathrm{d}\theta_{i,j})^{[2^j]}$ because $a_i \in F^{2^l}$ with $l > n$).

The left hand side of (37) is defined over $F$. For the right hand side, we compute the $F$-component of each term composing it. As a matter of fact:

(i) By Lemmas (3.5) and (3.6), and equation (38), the $F$-component of $a_1 \sum_{j=0}^{n-1} (\mathrm{d}\theta_{1,j})^{[2^j]} + a_2 \sum_{j=0}^{n-1} (\mathrm{d}\theta_{2,j})^{[2^j]}$ is contained in

$$\sum_{k=1}^{n} \sum_{j=1}^{2^{k-1}} b^{2j-1} (\mathrm{d}\Omega_F^{m-1})^{[2^k]} + \Omega_F^{m-1} \wedge \mathrm{d}b + \wp(\Omega_F^m) + \mathrm{d}\Omega_F^{m-1}.$$

(ii) The $F$-component of $\sum_{i=0}^{2^n-1} (\beta^{2i}\eta_i + \beta^{4i}\eta_i^{[2]})$ is obtained from the terms in the sum for which ($2^{n+1}$ divides $2i$) or ($2^{n+1}$ divides $4i$ but does not divide $2i$). In the first case, we collect elements of the shape $b^s\eta_i + b^{2s}\eta_i^{[2]}$ that belong to $\wp(\Omega_F^m) + \mathrm{d}\Omega_F^{m-1}$. In the second case, we necessarily have $i = 2^{n-1}$ (because $0 \leq i \leq 2^n - 1$), and thus we collect the element $\beta^{4i}\eta_i^{[2]} = b\eta_i^{[2]}$. For the other term in that sum, namely $\beta^{2i}\eta_i$, we see that this must cancel out with terms from the expression $a_1 \sum_{j=0}^{n-1} (\mathrm{d}\theta_{1,j})^{[2^j]} + a_2 \sum_{j=0}^{n-1} (\mathrm{d}\theta_{2,j})^{[2^j]} + \sum_{i=0}^{2^n-1} \beta^{2i}\mathrm{d}v_i$. Thus, $\eta_i$ belongs to $\sum_{j=0}^{n-1} (\mathrm{d}\Omega_F^{m-1})^{[2^j]}$ using equation (5). Therefore, $\beta^{4i}\eta_i^{[2]} = b\eta_i^{[2]}$ belongs to the group $\sum_{j=0}^{n-1} b(\mathrm{d}\Omega_F^{m-1})^{[2^{j+1}]} + \mathrm{d}\Omega_F^{m-1}$.

(iii) Finally, it is clear that the $F$-component of $\sum_{i=0}^{2^n-1} \beta^{2i}\mathrm{d}v_i$ is contained in $\mathrm{d}\Omega_F^{m-1}$.

Hence, we have proved that $w_L$ belongs to the group $\mathcal{J} := a_1 G_L^n(m) + a_2 G_L^n(m) + \sum_{k=1}^{n} \sum_{j=1}^{2^{k-1}} b^{2j-1} (\mathrm{d}\Omega_F^{m-1})^{[2^k]} + \Omega_F^{m-1} \wedge \mathrm{d}b + \wp(\Omega_F^m) + \mathrm{d}\Omega_F^{m-1}$. Since $\mathrm{Ker}(\Omega_F^m \longrightarrow \Omega_L^m) = \Omega_L^{m-1} \wedge \mathrm{d}b$, we conclude that $w \in \mathcal{J}$.

Conversely, Lemma 3.1 implies that the group $\overline{a_1 G_L^n(m)} + \overline{a_2 G_L^n(m)}$ is contained in the kernel $H_2^{m+1}(K \cdot L/F)$, and this kernel also contains the group $\sum_{k=1}^{n} \sum_{j=1}^{2^{k-1}} \overline{b^{2j-1} (\mathrm{d}\Omega_F^{m-1})^{[2^k]}} + \overline{\Omega_F^{m-1} \wedge \mathrm{d}b}$. This completes our proof. □

## 6. Examples about kernel splitting property

In this section we discuss the kernel splitting property when considering the compositum of purely inseparable extensions. The following proposition shows that purely inseparable multiquadratic extensions behave well with this property. More precisely we have:

**Proposition 6.1.** *Let $L/F$ be a finite purely inseparable extension (modular or not), and $K/F$ a multiquadratic purely inseparable extension of finite degree. Then*

$$H_2^{m+1}(K \cdot L/F) = H_2^{m+1}(K/F) + H_2^{m+1}(L/F).$$

*Proof.* Let $L = F(\sqrt[2^{n_1}]{c_1}, \ldots, \sqrt[2^{n_s}]{c_s})$ be a finite purely inseparable extension of $F$, where $c_1, \ldots, c_s \in F \setminus F^2$ and $n_1, \ldots, n_s$ positive integers. Let $K = F(\sqrt{c_{s+1}}, \ldots, \sqrt{c_{s+r}})$ be a multiquadratic purely inseparable extension of $F$, where $c_{s+1}, \ldots, c_{s+r} \in F \setminus F^2$. We have to prove that $H_2^{m+1}(K \cdot L/F) = H_2^{m+1}(L/F) + H_2^{m+1}(K/F)$.

Let $\rho = \max\{n_1, \ldots, n_s\}$ and $n_{s+1} = \cdots = n_{s+r} = 1$. According to the computation of the kernel $H_2^{m+1}(K \cdot L/F)$ [10], we know:

$$H_2^{m+1}(K \cdot L/F) = \sum_{i=1}^{s} H_2^{m+1}(F(\sqrt[2^{n_i}]{c_i})/F) + \sum_{i=s+1}^{s+r} H_2^{m+1}(F(\sqrt{c_i})/F) + S,$$

where

$$S = \sum_{k=1}^{\rho-1} \sum_{\substack{l_i 2^{n_i-k-1} \in \mathbb{N} \\ 0 \le l_i \le 2^k - 1}} \overline{c_1^{l_1} \cdots c_{r+s}^{l_{r+s}} (\mathrm{d}\Omega_F^{m-1})^{[2^k]}}.$$

Since $n_i = 1$ for any $i \in \{s+1, \ldots, s+r\}$, then $l_i$ satisfies both conditions: $0 \le l_i \le 2^k - 1$ and $l_i 2^{-k} \in \mathbb{N}$. Consequently, $l_i = 0$. This means that $\overline{S} \subset H_2^{m+1}(L/F)$. Therefore, $H_2^{m+1}(K \cdot L/F) = H_2^{m+1}(K/F) + H_2^{m+1}(L/F)$. □

We finish this section by constructing an example which shows that the kernel splitting property depends on the subfields of the compositum.

Let $k$ be a field of characteristic 2. We assume we can choose $b_1, b_2, b_3 \in k$ that are 2-independent over $k$ and let $F = k(t)$ be the rational function field in one indeterminate over $k$, and $K = F(\sqrt[4]{b_1}, \sqrt[4]{b_2}, \sqrt{b_3})$. Note that $K$ is a modular extension of $F$ because the elements $b_1, b_2, b_3$ are 2-independent over $F$ as they are 2-independent over $k$.

We consider $\mathcal{B} \subset F$ a 2-basis of $k$ containing $b_1, b_2, b_3$ and so $\mathcal{B} \cup \{t\}$ is an ordered 2-basis of $F$ in such a way that $t$ is the maximal element of it.

Defining $E_1 = F(\sqrt[4]{b_1}, \sqrt[4]{b_2})$ and $E_2 = F(\sqrt{b_3})$ then $K = E_1 \cdot E_2$. By Proposition 6.1

$$H_2^{m+1}(K/F) = H_2^{m+1}(E_1/F) + H_2^{m+1}(E_2/F), \text{ all } m \ge 1.$$

That is, we have a kernel splitting with respect to the subfields $E_1$ and $E_2$.

Now define $E_3 = F(\sqrt[4]{b_1}, \sqrt{b_3})$ and $E_4 = F(\sqrt[4]{b_2})$, clearly $K = E_3 \cdot E_4$. We claim the following:

**Example 6.2.**

$$H_2^2(K/F) \neq H_2^2(E_3/F) + H_2^2(E_4/F). \quad (39)$$

*Proof.* By [10, Theorems 4.1 and 3.1] we have

$$H_2^2(E_3/F) = \overline{b_1(\mathrm{d}F)^{[2]}} + \overline{F\mathrm{d}b_1} + \overline{F\mathrm{d}b_3},$$

and

$$H_2^2(E_4/F) = \overline{b_2(\mathrm{d}F)^{[2]}} + \overline{F\mathrm{d}b_2}.$$

Assume we have an equality in (39). Since $\overline{b_1b_2(\mathrm{d}F)^{[2]}} \subset H_2^2(K/F)$, then $\overline{b_1b_2(\mathrm{d}F)^{[2]}} \subset H_2^2(E_3/F) + H_2^2(E_4/F)$. Thus, for any $0 \neq \lambda \in F$ we must have

$$b_1b_2\,(\mathrm{d}\lambda)^{[2]} \in F\mathrm{d}b_1 + F\mathrm{d}b_2 + F\mathrm{d}b_3 + b_1\,(\mathrm{d}F)^{[2]} + b_2\,(\mathrm{d}F)^{[2]} + \wp(\Omega_F^1) + \mathrm{d}F \subseteq \Omega_F^1.$$

In particular choosing $\lambda = t$, there must exist $a_1, a_2, a_3, f_1, f_2, h_0 \in F, g \in \Omega_F^1$ such that

$$b_1b_2\,(\mathrm{d}t)^{[2]} \;=\; a_1\mathrm{d}b_1 + a_2\mathrm{d}b_2 + a_3\mathrm{d}b_3 + b_1\,(\mathrm{d}f_1)^{[2]} + b_2\,(\mathrm{d}f_2)^{[2]} + \wp(g) + \mathrm{d}h_0.$$

Computing modulo $\Omega^1_{F,<t}$, we get

$$b_1b_2t^2\frac{\mathrm{d}t}{t} = b_1\,(tD_t(f_1))^2\,\frac{\mathrm{d}t}{t} + b_2\,(tD_t(f_2))^2\,\frac{\mathrm{d}t}{t} + \wp(g)\frac{\mathrm{d}t}{t} + tD_t(h)\frac{\mathrm{d}t}{t}, \quad (40)$$

for some $h \in F$ (that takes care of the squaring process), here $D_t$ denotes the partial derivative with respect to $t$. Note that for $x \in F = k(t)$, $D_t(x)$ is a rational function on even powers of $t$.

Hence, we get from equation (40) the relation

$$b_1b_2t^2 + b_1\,(tD_t(f_1))^2 + b_2\,(tD_t(f_2))^2 + tD_t(h) = \wp(g). \quad (41)$$

Consequently, the 2-dimensional quadratic form

$$[1, b_1b_2t^2 + b_1\,(tD_t(f_1))^2 + b_2\,(tD_t(f_2))^2 + tD_t(h)]$$

is isotropic since its Arf invariant is trivial by (41). In particular, this implies that the quadratic form

$$\rho := [1, b_1b_2t^2] \perp \langle b_1, b_2, tD_t(h)\rangle \quad (42)$$

is isotropic.

Moreover, in equation (42), and modulo isometry, we may replace $D_t(h)$ by a polynomial $R \in k[t^{-2}]$ not divisible by $t^{-2}$. Consequently, the form $\langle b_1, b_2, tR\rangle$ is anisotropic because $\langle b_1, b_2\rangle$ is also anisotropic and $t^{-1}$ does not divide $R$.

Now the isotropy of $\rho$ and the uniqueness of the quasilinear part means the following isometry:

$$\rho := [0,0] \perp \langle b_1, b_2, tR\rangle \quad (43)$$

Let $\beta = \sqrt{b_1b_2}$ and $L = F(\beta)$. Note that $\langle b_1, b_2\rangle_L \simeq \langle 0\rangle \perp \langle b_1\rangle_L$ and $[1, b_1b_2t^2]_L \simeq [1, \beta t]$. Extending (43) to $L$ and canceling the form $\langle 0\rangle$, we deduce over $L$ the isometry

$$[1, \beta t] \perp \langle b_1, tR\rangle \simeq [0,0] \perp \langle b_1, tR\rangle\,.$$

Now the isotropy of $[1, \beta t] \perp \langle b_1, tR\rangle$ over $L$ means the existence of coprime polynomials $A, B, C, D \in k(\beta)[t^{-1}]$ not all zero such that

$$A^2 + AB + \beta t B^2 + b_1 C^2 + tRD^2 = 0. \tag{44}$$

Multiplying equation (44) by $t^{-1}$ and substuting $t^{-1}$ to $0$, we deduce that $t^{-1}$ divides $B$ and $D$, otherwise $\beta$ would be in $F$, which is not possible because $b_1, b_2$ are 2-independent. Writing $B = t^{-1}B'$ and $D = t^{-1}D'$ for suitable $B', D' \in k(\beta)[t^{-1}]$, and substituting in (44) we deduce

$$A^2 + t^{-1}AB' + \beta t^{-1}B'^2 + b_1 C^2 + t^{-1}RD'^2 = 0. \tag{45}$$

Now substituting $t^{-1}$ to 0 in (45) and using the fact that $\langle 1, b_1\rangle$ is anisotropic over $k(\beta)$ (because $b_1, b_2$ are 2-independent), we deduce that $A$ and $B$ are divisible by $t^{-1}$, which is a contradiction with the hypothesis that $A, B, C, D$ are coprime as polynomials of $k(\beta)[t^{-1}]$. □

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