MINICOURSE "ALGEBRAIC GROUPS AND HERMITIAN K-THEORY"

JUNE 18-22, 2018

I. Courses

1. Course of Stefan Gille

Title: Quadratic forms over semilocal rings.

Abstract: Quadratic forms over integers, rational-, real-, and/or complex numbers have been studied for a long time and one may trace back some of the discoveries even to ancient times. The modern algebraic theory of quadratic forms over a field started only with Ernst Witt's paper "Theorie der quadratischen Formen in beliebigen Körpern" published 1937 in Crelle's journal. Here in particular the Witt ring made its first appearance. But it was not before that mid 1960's that this theory started to flourish with the discovery of the Pfister forms and of connections to K-theory, a development which culminated in the famous Milnor conjectures. In these years mathematicians – partly driven by the usual desire for generalizations, partly by the hope this will give a better understanding of quadratic forms over fields – introduced and started to study Witt groups of rings (in particular local ones), varieties, and also of categories with duality.

In the first lecture of my course I will recall the definitions of quadratic and bilinear forms over rings respectively more general in categories with duality and their immediate consequences. The focus of the other four lectures will be on the following three topics:

1) Which properties of quadratic- and bilinear forms over a field hold also over a (semi-)local ring.

2) The Gersten conjecture and derived Witt groups.

3) Structure of the powers of the fundamental ideal of the Witt ring of a (semi-)local ring and connections with the Milnor conjecture.

2. Course of Ivan Panin

Title: On Grothendieck–Serre conjecture concerning principal bundles.

Abstract: Let R be a regular local ring. Let G be a reductive group scheme over R. A well-known conjecture due to Grothendieck and Serre assertes that a principal G-bundle over R is trivial, if it is trivial over the fraction field of R. In other words, if K is the fraction field of R, then the map of non-abelian cohomology pointed sets

$$H^1_{\text{ét}}(R, G) \longrightarrow H^1_{\text{ét}}(K, G),$$

induced by the inclusion of R into K, has a trivial kernel. If the ring R contains a field, then this conjecture is proved. This and other results concerning the conjecture will be discussed in these lectures. We illustrate the exposition by many interesting examples. We begin with couple results for complex algebraic varieties and develop the exposition step by step to its full generality.

3. Course of Marco Schlichting

Title: Introduction to Hermitian K-theory.

Abstract: An inner product space over a (commutative) ring R is a finitely generated projective R-module equipped with a non-degenerate symmetric bilinear form. Call two inner product spaces X, Y stably equivalent if there is a third such space Z and an isometry $X \perp Z \cong Y \perp Z$. The set of stable equivalence classes becomes an abelian monoid under orthogonal sum $\perp$ and embeds into the Grothendieck-Witt group $GW_0(R)$ of formal differences of stable equivalence classes. There are versions of the Grothendieck-Witt group for schemes and rings with involution.

The aim of the lectures is to extend the Grothendieck-Witt group to a bigraded cohomology theory GW_i^n for rings and schemes similar to Quillen's higher algebraic K-theory. The groups GW_i^n are defined using algebraic topology and their behaviour is governed by the structure of derived categories. We will explain definitions and the basic tools used to compute these groups, and make connections with algebraic K-theory, L-theory (triangular Witt groups), and topological K-theory.

II. Talks

1. Talk of Andrei Druzhinin

Title: The finite descent for Fr_*, $\widetilde{Cor}$, and GW correspondences.

Abstract: We extend the results on framed correspondences and framed motives introduced by V. Voevidsky and developed by G. Garkusha and I. Panin to the case of finite fields.

The results over finite fields can be deduced from the ones over (perfect) infinite fields because of so-called finite descent for framed correspondences. In the most precise form this means that there is a left inverse up to $\mathbf{A}^1$-homotopy and σ-suspensions and additivity relation to the morphism $S \to \operatorname{Spec} k$, $S = \operatorname{Spec} K_1 \amalg \operatorname{Spec} K_2$, K_1 and K_2 are extensions of a field k, $(\deg K_1, \deg K_2) = (\deg K_1, \operatorname{char} k) = (\deg K_2, \operatorname{char} k) = 1$.

For a given pair of correspondences over K_1 and K_2 this allows to define in a canonical way a correspondence over k, and on the categorical level this implies that $SH(k)$ $(SH^{fr}(k))$ is a full subcategory in $SH(\mathcal{S})$ over a simplicial scheme $\mathcal{S} = \{ \cdots S^i \cdots \Rrightarrow S^2 \rightrightarrows S \}$. So this is the descent with respect to the topology generated by a coverings $\operatorname{Spec} K_1 \amalg \operatorname{Spec} K_2 \to \operatorname{Spec} k$ as above.

The same construction gives finite descent for $\widetilde{Cor}(k)$ defined by B. Calmès, J. Fasel, and for GW correspondences. So consequently this yields the results for the theory of Milnor-Witt motives, and GW motives as well.

The result on framed motives over a finite fileds is part of a joint work with Jonas Irgens Kylling. The similar results ware obtained simultaneously also by Marc Hoyois, and Alexey Tsybyshev.

2. TALK OF SATYA MANDAL

Title: K **and** GW**-Theory of quasi-projective schemes.**

Abstract: Let X be a noetherian scheme. For $\mathcal{F} \in Coh(X)$, recall $grade(\mathcal{F}) := \min\{t : \mathcal{E}xt^t(\mathcal{F}, \mathcal{O}_X) \neq 0\}$. Note, $grade(\mathcal{F}) = co\dim(Supp(\mathcal{F}))$, if X is Cohen-Macaulay. Let $\mathscr{V}(X)$ denote the category of vector bundles on X. Define the subcategories

$$\begin{cases} \mathbb{M}^k(X) = \{\mathcal{F} \in Coh(X) : grade(\mathcal{F}) \geq k, \dim_{\mathscr{V}(X)}(\mathcal{F}) < \infty\} \\ C\mathbb{M}^k(X) = \{\mathcal{F} \in Coh(X) : grade(\mathcal{F}) = \dim_{\mathscr{V}(X)}(\mathcal{F}) = k\}. \end{cases}$$

When X is Cohen-Macaulay quasi projective, over an affine scheme, work of Schlichting is applied to derive localization theorems for K-theory and GW-theory relating the categories $C\mathbb{M}^{k+1}(X)$ and $C\mathbb{M}^k(X)$, and others. This is made possible by establishing the following sequence of equivalences of derived categories:

$$\mathcal{D}^b(C\mathbb{M}^k(X)) \xrightarrow[\zeta]{\sim} \mathcal{D}^b(\mathbb{M}^k(X)) \xrightarrow[\iota]{\sim} \mathscr{D}^k(\mathbb{M}^0(X)) \xleftarrow[\iota']{\sim} \mathscr{D}^k(\mathscr{V}(X)) \tag{1}$$

where $\mathcal{D}^b(-)$ denotes the bounded derived category, and $\mathscr{D}^k(-) \subseteq \mathcal{D}^b(-)$ denotes the filtration induced by the grade of the homologies. Key to the proof of (1) is a construction of a chain complex map, due to Hans-Bjørn Foxby, to a given complex, from a Koszul complex, to "approximate" homologies.

Note, $C\mathbb{M}^k(X)$ has a natural duality, and hence a GW-theory is viable, contrary to $\mathbb{M}^k(X)$ and others.

3. TALK OF DENIS NARDIN

Title: A universal property for Hermitian K-theory

Abstract: In this talk we construct a category of hermitian noncommutative motives and we prove that the hermitian K-theory functor is corepresentable by the noncommutative motive of the sphere spectrum. The approach taken clarifies the relation between the different definitions of hermitian K-theory and L-theory in the literature when 2 is not invertible.

4. TALK OF HUSNEY PARVEZ SARWAR

Title: K-Theory of monoid algebras and questions of Gubeladze.

Abstract: We will begin with some questions of Gubeladze on K-theory of monoid algebras. The question is whether or not the homotopy invariance property of K-theory extends to the monoid extension. We will present some results in this direction. At the end we will also present a result which is monoid analog of Weibel's vanishing conjecture. (This is a joint work with Amalendu Krishna).

5. TALK OF HENG XIE

Title: Hermitian K-theory of schemes with involution.

Abstract: : Hermitian K-theory of schemes with involution can be viewed as an algebraic generalization of Atiyah's topological KR-theory in 1960's. In this short talk, I will prove a version of the dévissage theorem for Hermitian K-theory of schemes with involution. It provides a tool for us to compute Hermitian K-theory of schemes with involution by invariant closed subschemes. If time is permitted, I will compute Hermitian K-theory of the projective line P^1 with the involution given by switching coordinates. This computation yields the C_2-equivariant A^1-invariance for Hermitian K-theory by the dévissage theorem.